V. Probability
(Part II): Intermediate Topics

- Joint, Marginal and Conditional Probabilites
- Total Probability, Bayes' Theorem
- Mathematical Expectation

Joint, Marginal and Conditional Probabilities (1/3)

Example: **Joint Probability**

A rating agency keeps track of 500 stocks. 300 are from New York and 200 from Frankfurt. Two hundred are up, 100 are unchanged, and 200 are down. The results are shown in the table below:

Stock Exchange	Up	Unchanged	Down	Total
NYSE	50	75	175	300
Frankfurt	150	**25**	25	200
Total	200	100	200	**500**

Joint Probability

Problem: Find the joint probability that one randomly selected paper is from Frankfurt and unchanged.
Solution:

$$\mathrm{P}(\text{Frankfurt} \wedge \text{unchanged}) = \frac{25}{500} = 5\%$$

Joint, Marginal and Conditional Probabilities (2/3)

Example: **Marginal Probability**

A rating agency keeps track of 500 stocks. 300 are from New York and 200 from Frankfurt. Two hundred are up, 100 are unchanged, and 200 are down. The results are shown in the table below:

Stock Exchange	Up	Unchanged	Down	Total
NYSE	50	75	175	300
Frankfurt	150	25	25	**200**
Total	200	100	200	**500**

Marginal Probability

Problem: Find the marginal probability that one randomly selected paper is from Frankfurt.
Solution:

$$\mathrm{P}(\text{Frankfurt}) = \frac{200}{500} = 40\%$$

Joint, Marginal and Conditional Probabilities (3/3)

Example: **Conditional Probability**

A rating agency keeps track of 500 stocks. 300 are from New York and 200 from Frankfurt. Two hundred are up, 100 are unchanged, and 200 are down. The results are shown in the table below:

Stock Exchange	Up	Unchanged	Down	Total
NYSE	50	75	175	300
Frankfurt	150	**25**	25	**200**
Total	200	100	200	500

Conditional Probability

Problem: Find the conditional probability that one randomly selected paper is unchanged, given that is from Frankfurt.
Solution: We have to use the a priori information that the paper is from Frankfurt.

$$\mathrm{P}(\text{unchanged} \mid \text{Frankfurt}) = \frac{25}{200} = 12.5\%$$

Total Probability, Bayes' Theorem (1/9)

Consider the following example: A car manufacturer receives the air-conditioning systems from three different suppliers. 50% are from supplier B_1, 30% from supplier B_2 and 20% from B_3. Unfortunately, 5% of the delivered air cons from supplier B_1 are of inferior quality. For supplier B_2 and B_3 these percentages are 9% and 24%, respectively.

Example

supplier	% of air cons	% inferior quality
supplier B_1	50%	5%
supplier B_2	30%	9%
supplier B_3	20%	24%
	100%	

We can ask the following questions:

- What ist the probability that a randomly selected car has a defective air condition? **(total probability)**
- Given the fact that the air condition does not work, what is the probability that it is from supplier B_1 (or B_2, or B_3)? **(Bayes' theorem)**

Total Probability, Bayes' Theorem (2/9)

The Law of Total Probability

If $\{B_n : n \in 1, \cdots, k\}$ is a (finite) partition of a sample space (a set of **pairwise disjoint** events) and

$$\sum_{n=1}^{k} B_n = 1$$

then for any event A the following theorem:

$$P(A) = \sum_{n=1}^{k} P(B_k) \cdot P(A|B_k)$$

holds.

Total Probability, Bayes' Theorem (3/9)

Let us find an answer to the first question:

What ist the probability that a randomly selected car has a defective air condition?

supplier	% of air cons	% inferior quality
supplier B_1	50%	5%
supplier B_2	30%	9%
supplier B_3	20%	24%
	100%	

Total Probability

Problem: We want to find the **total probability.** For simplicity, let A denote the event „defective" (of inferior quality in the table)
Solution:

$$\text{P(A)} = \underbrace{P(B_1)}_{0.5} \cdot \underbrace{P(A|B_1)}_{0.05} + \underbrace{P(B_2)}_{0.3} \cdot \underbrace{P(A|B_2)}_{0.09} + \underbrace{P(B_3)}_{0.2} \cdot \underbrace{P(A|B_3)}_{0.24} = 0.1 = \underline{10\%}$$

Total Probability, Bayes' Theorem (4/9)

Please Note:

$$P(B|A) \neq P(A|B)$$

Bayes' Theorem [a]

$$P(B_k|A) = \frac{P(B_k) \cdot P(A|B_k)}{\sum_{n=1}^{k} P(B_k) \cdot P(A|B_k)}$$

The theorem holds if $P(A) \neq 0$. Note that the expression in the denominator is the same as P(A) according to the **law of total probability**.

[a] Thomas Bayes, English statistician, philosopher, 1701 - 1761

Total Probability, Bayes' Theorem (5/9)

Let us find an answer to the second question:

Given the fact that the air condition does not work, what is the probability that it is from supplier B_1 (or B_2, or B_3)?

supplier	% of air cons	% inferior quality
supplier B_1	50%	5%
supplier B_2	30%	9%
supplier B_3	20%	24%
	100%	

Bayes' Theorem

Problem: We want to find the Probability $P(B_1|A)$ (Again, note that this is **not** the same as $P(A|B_1)$!)
Solution:

$$P(B_1|A) = \frac{P(B_1) \cdot P(A|B_1)}{\sum_{n=1}^{k} P(B_k) \cdot P(A|B_k)} = \frac{0.5 \cdot 0.05}{0.1} = \underline{25\%}$$

If we have a defective air condition, the probability that it is from supplier B_1 equals 25%.

Total Probability, Bayes' Theorem (6/9)

supplier	% of air cons	% inferior quality
supplier B_1	50%	5%
supplier B_2	30%	9%
supplier B_3	20%	24%
	100%	

Example (continued)

We already calculated $P(A) = 10\%$ and $P(B_1|A) = 25\%$. Use these results to find ...

1. ...the probability that a defective air condition is from supplier B_2
2. ...the probability that a defective air condition is from supplier B_3

Total Probability, Bayes' Theorem (7/9)

supplier	% of air cons	% inferior quality
supplier B_1	50%	5%
supplier B_2	30%	9%
supplier B_3	20%	24%
	100%	

Solution:

We already calculated $P(A) = 10\%$ and $P(B_1|A) = \underline{25\%}$ use these results to find ...

1. $P(B_2|A) = \frac{0.3 \cdot 0.09}{0.1} = \underline{27\%}$
2. $P(B_3|A) = \frac{0.2 \cdot 0.24}{0.1} = \underline{48\%}$

This really makes sense! If we have a product of ineferior quality and three suppliers, it is either from B_1 *or* from B_2 *or* from B_3!
(25% + 27% + 48% = 100%)

Total Probability, Bayes' Theorem (8/9)

Example: Lottery Tickets

In a lottery game the tickets are kept in different boxes. Box 1 contains 100 tickets (30 of them win), box 2 contains 100 tickets (50 of them win) and box 3 also 100 tickets (but only 25 of them win). All three boxes are emptied into a larger container, shuffled and one ticket is drawn at random.

Try to find a solution to the following problems:

1. What is the total probability to win a prize?
2. Given it is a winning ticket, what is the probability it was initially in ...
   - 2.1 ...box 1?
   - 2.2 ...box 2?
   - 2.3 ...box 3?

Total Probability, Bayes' Theorem (9/9)

Solution: Lottery Tickets

Urn	proportion of tickets	winning probability
B_1 (Urn 1)	1/3	30%
B_2 (Urn 2)	1/3	50%
B_3 (Urn 3)	1/3	25%

Let A denote the event that it is a winning ticket.

1. $P(A) = P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) + P(B_3) \cdot P(A|B_3) =$
   $P(A) = 1/3 \cdot 0.3 + 1/3 \cdot 0.5 + 1/3 \cdot 0.25 = \underline{35\%}$
2. Given it is a winning ticket, what is the probability it was initially in ...
   2.1 ... box 1: $P(B_1|A) = \frac{P(B_1) \cdot P(A|B_1)}{P(A)} = \frac{1/3 \cdot 0{,}3}{0{,}35} \approx \underline{28{,}57\%}$
   2.2 ... box 2: $P(B_2|A) = \frac{P(B_2) \cdot P(A|B_2)}{P(A)} = \frac{1/3 \cdot 0{,}5}{0{,}35} \approx \underline{47{,}62\%}$
   2.3 ... box 3: $P(B_3|A) = \frac{P(B_3) \cdot P(A|B_3)}{P(A)} = \frac{1/3 \cdot 0{,}25}{0{,}35} \approx \underline{23{,}81\%}$

Mathematical Expectation (1/5)

Mathematical Expectation

If X denotes a **discrete** random variable that can assume the values $X_1, X_2, \cdots, X_k$ with respective probabilities $p_1, p_2, \cdots, p_k$, where $p_1 + p_2 + \cdots + p_k = 1$, the (mathematical) expectation of X, denoted by $E(X)$, is defined as

$$E(X) = \sum_{i=1}^{n} p_i X_i$$

If X denotes a **continuous** random variable and $f(x)$ is the value of its probability distribution at x, the expected value is defined as

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

Mathematical Expectation (2/5)

Rules for Mathematical Expectation

For random variables X and Y and for constants values b and c the following theorems hold:

- $E(c \cdot X) = c \cdot E(x)$
- $E(X + Y) = E(X) + E(Y)$
- $E(X + b) = E(X) + b$

Fair Games

A **fair game** is one in which the cost of playing the game equals the expected winnings of the game, so that net value of the game equals zero. Note that - in a casino - games are usually **not** fair. You will come across this term also to decision theory - its usage is not restricted to gambling problems.

Mathematical Expectation (3/5)

Example 1: Business Venture

In a given business venture a businessman can make a profit of €300 with probability 60% or take a loss of €100 with probability 40% Determine his expectation.
Solution: €140

Example 2: Lottery Ticket

If a man purchases a lottery ticket, he can win a first prize of €5000 or a second prize of €2000 with probabilities 0.001 and 0.003. What should be a **fair** price to pay for the ticket?
Solution: €11

Mathematical Expectation (4/5)

Example 3: Casino

In a roulette game, players may choose to place bets on either a single number, various groupings of numbers, the colors red or black, whether the number is odd or even, or if the numbers are high or low. 18 numbers are black, 18 are red and the 0 is green. If one places a bet on a color and wins, the payoff equals twice the bet.

If you place bets on either red or black (say, 10 €) what is the mathematical expectation? What is your expected value if you decide to play 80 games (10 € per game)? How much money would you expect to lose/win?

Mathematical Expectation (5/5)

Example 3: Casino (Solution to the Problem)

We get the following events with their respective probabilities:

x_i	p_i
$x_1 = +10$ €	$p_1 = \frac{18}{37}$
$x_2 = -10$ €	$p_2 = \frac{19}{37}$

We can calculate the expected value:

$$E(X) = \sum_{i=1}^{n} p_i X_i = \frac{18}{37} \cdot 10 + \frac{19}{37} \cdot (-10) \approx -0.2702703$$

If we play 80 times, we get $80 \cdot (-0.2702703) \approx -21.62162$ We expect to lose 21.62 €.

VI. Probability (Part III: Discrete Probability Distributions)

- Binomial Distribution
- Hypergeometric Distribution
- Poisson Distribution

Binomial Distribution (1/5)

The binomial distribution is the discrete probability distribution of the number of successes in a sequence of n independent experiments, the outcome of each experiment is either success or failure (with probabilities p and 1-p).

- n independent trials
- success with probability p, failure with probabity 1-p
- The probability of success is exactly the same from one trial to another

Binomial Distribution

$$P(\text{k successes of n trials}) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

Binomial Distribution (2/5)

The Formula Explained

Problem: A biased coin is flipped three times. The probability for head is 70 %. Find the probability that head occurs exactly twice (2 successes out of 3 trials).
Solution: Without the formula we could use of a tree diagram and tabulate the probabilities for each sequence of results that give exactly two successes:

sequence	probability	$\cdots$ is the same as $\cdots$	equals
HHT	$P(HHT) = 0.7 \cdot 0.7 \cdot 0.3$	$0.7^2 \cdot 0.3^1$	0.147
HTH	$P(HTH) = 0.7 \cdot 0.3 \cdot 0.7$	$0.7^2 \cdot 0.3^1$	0.147
THH	$P(THH) = 0.3 \cdot 0.7 \cdot 0.7$	$0.7^2 \cdot 0.3^1$	0.147
		SUM	**0.441**

The probability that head occurs exactly twice equals **44.1 %**.
We have three sequences (HHT, HTH, THH) that lead to the desired result (H occurs twice). Each line in the table has the same probability $p^k \cdot (1-p)^{n-k}$ (here: 0.147). We simply have to mulitply this probability with the number of lines in the table. The number of lines is $\binom{n}{k}$ - here: $\binom{3}{2} = 3$.

Binomial Distribution (3/5)

EXCEL-Functions

=COMBIN(n;k)	Gives the number of combinations, i. e. $\binom{n}{k}$
=BINOM.DIST(k;n;p;0)	Gives the probability $P(X = k)$
=BINOM.DIST(k;n;p;1)	Gives the cumulated probability $\sum_{i=0}^{k} P(X = i)$

Biased Coin in Excel

A biased coin is flipped three times. The probability for head is 70 %. Find the probability that head occurs exactly twice (2 successes out of 3 trials).

Biased coin...

	A	B	C
1	k (head)	n-k (tail)	probability
2	0	3	**0,027**
3	1	2	**0,189**
4	2	1	**0,441**
5	3	0	**0,343**
6			

Biased coin (formula view)

	A	B	C
1	k (head)	n-k (tail)	probability
2	0	=3-A2	**=COMBIN(3;A2)*0,7^A2*0,3^B2**
3	1	=3-A3	**=COMBIN(3;A3)*0,7^A3*0,3^B3**
4	2	=3-A4	**=COMBIN(3;A4)*0,7^A4*0,3^B4**
5	3	=3-A5	**=COMBIN(3;A5)*0,7^A5*0,3^B5**

Binomial Distribution (4/5)

Example

Problem
From market research we found out that 70% of people who purchase a certain product are men. If 10 owners are randomly selected, find the probability that exactly 8 are men.

Solution
Let k denote the number of males. Possible values are $\Omega_k = \{0, 1, 2, \cdots, 10\}$. Use the formula $P(\text{k=8 men}) = \binom{10}{8} \cdot 0.7^8 \cdot 0.3^2 = 0.2334744 \approx$ **23.35**%

Binomial Distribution (5/5)

Example 1

Problem
80% percent of companies in the European Union use social media to recruit workers. Find the probability that in a survey of ten companies exactly half of them use social media.
(Solution: ≈ 2.64 %)

Example 2

Problem
A balanced coin is tossed 10 times, and the number of times a head occurs is represented by X. Find the probability that $X \geq 8$. (Solution: ≈ 5.47 %)

Hypergeometric Distribution (1/3)

The classical application of the hypergeometric distribution is sampling **without replacement**.

- **n** trials (draws)
- from a population with **N** elements
- in the population **K** elements have a certain property („success“), and **N**-**K** do not have this property
- We can calcualte the probability that exactly **k** out of n trials are successes - (if $0 \leq k \leq n$).

Hypergeometric Distribution (2/3)

- n number of draws
- N is the population size
- K is the number of success states in the population
- k is the number of observed successes

$$P(X = k) = \begin{cases} \frac{\binom{K}{k} \cdot \binom{N-K}{n-k}}{\binom{N}{n}} & \text{if } 0 \leq k \leq K \\ 0 & \text{else.} \end{cases}$$

Hypergeometric Distribution (3/3)

Example:

A retailer has 10 identical laptops of a company out which 4 are defective. If 3 laptops are selected at random (**without replacement!**), Find the probability that exactly 2 are defective.

Solution:

- n number of draws **n=3**
- N is the population size **N=10**
- K is the number of success states in the population **K=4**
- k is the number of observed successes **k=2**

Possible values for k: $\Omega_k = \{0, 1, 2, 3\}$.

$$P(k = 2) = \frac{\binom{4}{2} \cdot \binom{6}{1}}{\binom{10}{3}} = 30\%$$

Poisson Distribution (1/2)

- Discrete probability distribution
- **Approximation for the binomial distribution if p is small and n is large. The smaller p and the larger n, the better the approximation!**
- The Poisson distribution is popular for modelling the number of times an event occurs in an interval of time or space.

Poisson Distribution

$$P(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

Where

- λ is the average number of events
- e is Euler's number ($e \approx 2.718282$)
- k is the number of events, $k \in \{0, 1, 2 \ldots\}$

Poisson Distribution (2/2)

Example:

Suppose that the number of typing errors on a single page of a novel has a Poisson distribution with parameter $\lambda = \frac{1}{2}$. Find the probability that

- There is no error on a page.
- There is exactly one error on a page.

Solution:

- $P(k = 0) = e^{-\frac{1}{2}} \cdot \frac{(\frac{1}{2})^0}{0!} \approx 60.65\%$
- $P(k = 1) = e^{-\frac{1}{2}} \cdot \frac{(\frac{1}{2})^1}{1!} \approx 30.33\%$