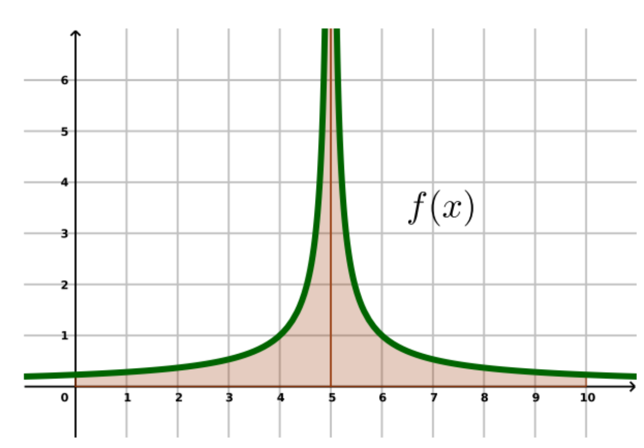
Worksheet 9
Example 4

$$f(x) = \frac{1}{\sqrt[11]{(x-5)^{10}}}, \quad [0, 10]$$

$$A = ?$$



$$\int \frac{1}{\sqrt[11]{(x-5)^{10}}}\, dx$$

$\left\{ \text{substitute } u = x-5, \quad du = dx \right\}$

$$\int \frac{1}{\sqrt[11]{u^{10}}}\, du =$$

$$\int u^{-10/11}\, du = \frac{u^{-\frac{10}{11}+1}}{\left(\frac{1}{11}\right)} + c = 11 \cdot u^{1/11} + c = 11 \cdot \sqrt[11]{u} + c = \underbrace{11 \cdot \sqrt[11]{x-5} + c}$$

We consider $x \geq 5$ and then use symmetry

$$\lim_{b \to 5^+} \int_b^{10} \frac{1}{\sqrt[11]{x-5}^{10}}\, dx = \lim_{b \to 5^+} \left(11 \sqrt[11]{x-5}\right)\Big|_b^{10}$$

$$= \lim_{b \to 5^+} 11 \left(\underbrace{\sqrt[11]{10-5}} - \sqrt[11]{b-5}\right)$$

$$= 11 \cdot \left(\sqrt[11]{5} - 0\right) = 11 \cdot \sqrt[11]{5}$$

$$A = 2 \cdot 11 \cdot \sqrt[11]{5} = 22 \cdot \sqrt[11]{5}$$

Example 5

Volume of $f(x) = \sin(\pi x)$ rotates around x-axis

$x \in [0, 9]$

$\left\{ \sin^2\vartheta = \frac{1-\cos 2\vartheta}{2} \right.$

$$V = \pi \cdot \int_0^1 (f(x))^2\, dx = \pi \int_0^1 \sin^2(\pi x)\, dx$$

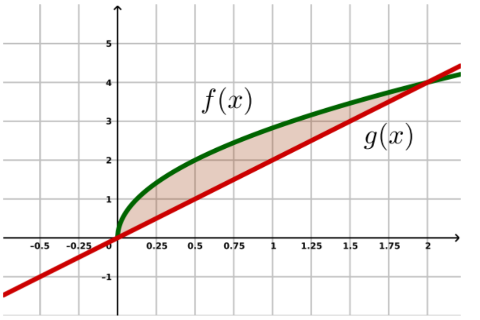
$$= \pi \int_0^1 \left(\frac{1-\cos(2\pi x)}{2}\right) dx$$

$$= \frac{\pi}{2} \int_0^1 1 - \cos(2\pi x)\, dx$$

$$= \frac{\pi}{2}\left(x\Big|_0^1 - \frac{\sin(2\pi x)}{2\pi}\Big|_0^1 \right)$$

$$= \frac{\pi}{2}\left(1 - \frac{\sin(2\pi)}{2\pi}\right) = \frac{\pi}{2}$$

Example 6



$$f(x) = \sqrt{8x}, \quad g(x) = 2x$$

Intersection points

$$f(x) = g(x)$$
$$\sqrt{8x} = 2x$$
$$(\sqrt{8x})^2 = (2x)^2$$
$$8x = 4x^2$$
$$4x^2 - 8x = 0$$
$$4x(x-2) = 0 \Rightarrow \begin{cases} x = 0 \\ x = 2 \end{cases}$$

$$\pi \int_0^2 f^2(x) - g^2(x)\, dx =$$

$$\pi \int_0^2 (\sqrt{8x})^2 - (2x)^2\, dx =$$

$$\pi \int_0^2 8x - 4x^2\, dx =$$

$$\pi \left(8\frac{x^2}{2}\Big|_0^2 - 4\frac{x^3}{3}\Big|_0^2 \right) =$$

$$\pi\left(4 \cdot 4 - \frac{4}{3} \cdot 8\right) = 16\pi - \frac{32}{3}\pi = 16.76$$

Example 2

- $\int_1^5 \frac{x}{\sqrt{x^2-1}}\,dx =$ substitution $x^2-1=u$

$2x\,dx = du$

$x\,dx = \frac{1}{2}du$

$$\int_1^5 \frac{1}{2\sqrt{u}}\,du = \frac{1}{2}\int_1^5 \frac{1}{\sqrt{u}}\,du$$

$$= \frac{1}{2}\int_1^5 u^{-1/2}\,du$$

$$= \frac{1}{\cancel{2}}\cdot\cancel{2}\cdot u^{1/2}\Big|_1^5 = \sqrt{u}\Big|_1^5 = \sqrt{x^2-1}\Big|_1^5$$

$$= \sqrt{5^2-1} = \sqrt{24} = 2\sqrt{6}$$

- $\int_1^2 \frac{1}{x-1}\,dx = ?$

$$\int \frac{1}{x-1}\,dx = \ln(x-1)$$

$$\lim_{\alpha\to 1^+}\Big[\ln(x-1)\Big]_\alpha^2 = \lim_{\alpha\to 1^+}\left[\ln(2-1)-\ln(\alpha-1)\right]$$

$$= \lim_{\alpha\to 1^+}\left[\underbrace{\ln 1}_{0}-\ln(\alpha-1)\right]$$

$$= 0 - \lim_{\alpha\to 1^+}\ln(\alpha-1)$$

$$= -\ln 0 = +\infty$$

$\alpha \to 1^+$

$\alpha - 1 \to 0$

Worksheet 10

example 1

$$A = \int_D 1 \, dA$$

$$D = \{(x, y) : 0 \le x \le \pi \text{ and } 2x\cdot(x-\pi) \le y \le 3\cdot\sin x\}$$

$$A = \int_0^{\pi} \int_{2x(x-\pi)}^{3\sin x} \; dy \, dx = \int_0^{\pi} y \Big|_{2x(x-\pi)}^{3\sin x} dx$$

$$= \int_0^{\pi} \left[3\sin x - 2x(x-\pi)\right] dx$$

$$= \int_0^{\pi} 3\sin x - 2x^2 + 2x\pi \, dx$$

$$= -3\cos x - \frac{2x^3}{3} + 2\pi\frac{x^2}{2} \Big|_0^{\pi} \approx \ldots \approx 16.335$$