Exercise Sheet 10: Integral in Two or more Dimensions

Example 1 *Let the function $f(x,y) = xy$ and the domain D (see figure 1) be given. Compute*

$$\int\limits_D f \, \mathrm{d}A$$

a) exactly using Fubini,

b) numerically using the midpoint rule. Use the mesh of figure 2. You do not have to compute it manually. It is enough, if you understand the computation of file "midpointrule.ods".

c) Visit the homepage `https://freefem.org/tryit` *and type in the following code. Play with n to change the accuracy, see figure 3.*

```
/* Integrate f = x * y
   in domain 0 <= x <= 1 and 0 <= y <= 1 - x^2 */

int n = 5; // Accuracy

// Boundary parts
border bottom(t = 0, 1) {x = t; y = 0; label = 1;}
border curve (t = 0, 1) {x = 1-t; y = 1 - (1-t)*(1-t); label = 2;}
border left  (t = 0, 1) {x = 0; y = 1 - t; label = 3;}

// Generate mesh
// n intervals for bottom bottom
// 4*n intervals for curve
// n intervals for left
mesh Th = buildmesh(bottom(n) + curve(4*n) + left(n));

// Compute integral
real I = int2d(Th)(x*y);

cout << "Integral = " << I << endl;

// Plot mesh
plot(Th, wait=true);
```

Solution: $\frac{1}{12} = 0{,}08\dot{3}$*, see Excel sheet, see homepage*

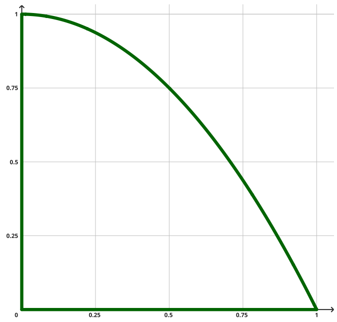

Figure 1: Gebiet

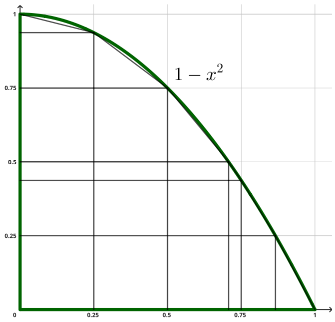

Figure 2: Mesh

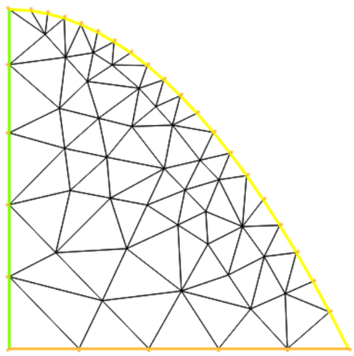

Figure 3: Mesh

Example 2 *Compute the area*

$$A = \int_D 1 \,\mathrm{d}A,$$

where

$$D = \{(x,y) \colon 0 \leq x \leq \pi \text{ and } 2x \cdot (x-\pi) \leq y \leq 3 \cdot \sin x\}.$$

A sketch of D can be helpful!

Next compute the "mass", if the density is given by $\varrho \equiv 2$.

$$m = \int_D \varrho \,\mathrm{d}A.$$

Find the centre of mass $S = (S_x, S_y)$? Use the formulas:

$$S_x = \frac{1}{m} \cdot \int_D x \cdot \varrho \,\mathrm{d}A \quad S_y = \frac{1}{m} \cdot \int_D y \cdot \varrho \,\mathrm{d}A.$$

If necessary, use integration by parts and Pythagoras' theorem to compute

$$\int \sin^2(x) \,\mathrm{d}x = \frac{x}{2} - \frac{\sin(x) \cdot \cos(x)}{2} + c.$$

Or you can use this result without derivation (it is almost the same as on a previous exercise sheet).

Solution:
$A = 6 + \frac{\pi^3}{3} = 16.335\ldots$, $m = 32.670\ldots$, $S = \left(\frac{\pi}{2}, -0.816\ldots\right)$

Example 3 *Compute the average temperature of the annulus*

$$9 \leq x^2 + y^2 \leq 16,$$

where the density of the temperature is given by $t(x,y) = \sqrt{x^2+y^2}$. *Use the formula*

$$\bar{t} = \frac{1}{A} \cdot \int\limits_D t \, \mathrm{d}A.$$

Solution:
$A = \frac{7\pi}{2} = 10.995\ldots$, $\bar{t} = \frac{74}{21} = 3.523\ldots$

Example 4 *The American track and field athlete Dick Fosbury revolutionized the high jump with the so-called Fosbury Flop, see figure 4. We model the body of the athlete in*

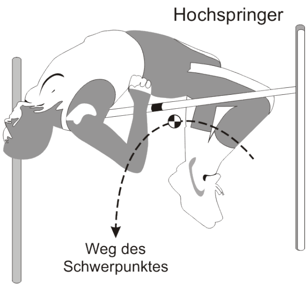

Figure 4: Fosbury Flop.

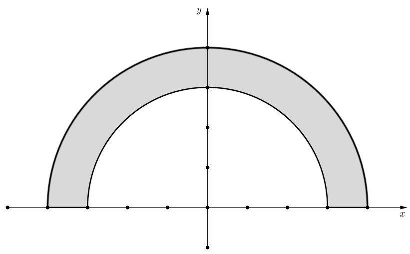

Figure 5: Kreisring.

the plain by the simple annulus $9 \leq x^2 + y^2 \leq 16$ *where* $y \geq 0$. *The annulus (see figure 5) is assumed to have a homogeneous mass density, i.e.* $\varrho(x,y) \equiv 1$. *Compute the centre of mass* $S = (S_x, S_y)$ *of the annulus and discuss its consequence!*

Solution:
$m = \frac{7\pi}{2} = 10.99\ldots$, $S = \left(0, \frac{148}{21\pi}\right) = (0, 2.24\ldots)$ *and therefore below the bar.*

Example 5 *Make a sketch of the curve* $r(\varphi) = \sqrt{3\cos(\varphi)}$ *in* φ-r-*plane and* x-y-*plane, where* $\varphi \in \left[0, \frac{\pi}{2}\right]$ *and compute the area for the corresponding domain, i.e.* $r \in [0, r(\varphi)]$ *and* $\varphi \in \left[0, \frac{\pi}{2}\right]$.

Solution:
$A = \frac{3}{2}$

Example 6 *Compute the integral*

$$I = \int\limits_{x=0}^{2} \int\limits_{y=-1}^{2} \int\limits_{z=0}^{\pi} x^2 y \sin(yz) \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x.$$

Solution:

$$I = \int\limits_{x=0}^{2}\int\limits_{y=-1}^{2}\int\limits_{z=0}^{\pi} x^2 y \sin(yz)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int\limits_{x=0}^{2}\int\limits_{y=-1}^{2} x^2 y\,(-\cos(yz))\,\frac{1}{y}\bigg|_{z=0}^{\pi}\,\mathrm{d}y\,\mathrm{d}x$$

$$= -\int\limits_{x=0}^{2}\int\limits_{y=-1}^{2} x^2\cos(yz)\big|_{z=0}^{\pi}\,\mathrm{d}y\,\mathrm{d}x = -\int\limits_{x=0}^{2}\int\limits_{y=-1}^{2}\left(x^2\cos(\pi y) - x^2\right)\,\mathrm{d}y\,\mathrm{d}x$$

$$= -\int\limits_{x=0}^{2} x^2\sin(\pi y)\frac{1}{\pi} - x^2 y\bigg|_{y=-1}^{2}\,\mathrm{d}x$$

$$= -\int\limits_{x=0}^{2}\left(x^2\sin(2\pi)\frac{1}{\pi} - 2x^2 - (x^2\sin(-\pi) + x^2)\right)\,\mathrm{d}x$$

$$= -\int\limits_{x=0}^{2}\left(0 - 2x^2 - (0 + x^2)\right)\,\mathrm{d}x = \int\limits_{x=0}^{2} 3x^2 dx = x^3\big|_{x=0}^{2} = 8 - 0 = 8$$

Example 7 *Compute the integral*

$$I = \int\limits_{x=0}^{\frac{\pi}{2}}\int\limits_{y=0}^{1}\int\limits_{z=y}^{y^2} yz\cos(x)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Solution:

$$I = \int\limits_{x=0}^{\frac{\pi}{2}}\int\limits_{y=0}^{1}\int\limits_{z=y}^{y^2} yz\cos(x)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2}\int\limits_{x=0}^{\frac{\pi}{2}}\int\limits_{y=0}^{1}\left[yz^2\cos(x)\right]_{z=y}^{y^2}\,\mathrm{d}y\,\mathrm{d}x$$

$$= \frac{1}{2}\int\limits_{x=0}^{\frac{\pi}{2}}\int\limits_{y=0}^{1}(y^5\cos(x) - y^3\cos(x))\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2}\int\limits_{x=0}^{\frac{\pi}{2}}\left[\frac{y^6}{6}\cos(x) - \frac{y^4}{4}\cos(x)\right]_{y=0}^{1}\,\mathrm{d}x$$

$$= \frac{1}{2}\int\limits_{x=0}^{\frac{\pi}{2}}\left(\frac{1}{6}\cos(x) - \frac{1}{4}\cos(x) - 0\right)\,\mathrm{d}x = \frac{1}{2}\left[\frac{1}{6}\sin(x) - \frac{1}{4}\sin(x)\right]_{x=0}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{1}{6} - \frac{1}{4} - 0\right) = \frac{1}{2}\left(\frac{-1}{12}\right) = -\frac{1}{24}$$

Example 8 *Compute the volume, mass and centre of mass of of the cube* $G = [0,1]^3$*,* *if the density is given by* $\varrho(x, y, z) = 1 + x + y + z$. *Use the formulas:*

$$V = \int\limits_D 1 \,\mathrm{d}V, \quad m = \int\limits_D \varrho \,\mathrm{d}V,$$

$$S_x = \frac{1}{m} \cdot \int\limits_D x \cdot \varrho \,\mathrm{d}V, \quad S_y = \frac{1}{m} \cdot \int\limits_D y \cdot \varrho \,\mathrm{d}V, \quad S_z = \frac{1}{m} \cdot \int\limits_D z \cdot \varrho \,\mathrm{d}V$$

Solution:
$V = 1,\ m = 2,5,\ S = \frac{8}{15} \cdot (1, 1, 1) = (0.533\ldots, 0.533\ldots, 0.533\ldots)$