Integrals

Repetition Integrals in One Dimension

Antiderivative

For $F(x) = x^2 + \sin(x) \xrightarrow{\text{Differentiation}} F'(x) = 2x + \cos(x)$

$G(x) = x^2 + \sin(x) + 7 \xrightarrow{\text{Differentiation}} G'(x) = 2x + \cos(x)$

Integration

$F(x)$ is antiderivative of $2x+\cos(x)$ but also $G(x)$

Up to a constant c the antiderivative is unique and we use notation:

$$\int 2x+\cos(x)\,dx \quad \text{for the antiderivatives}$$

$$\shortparallel$$

$$x^2+\sin(x)+c$$

$$\begin{tikzcd}
F \arrow[r, bend left, "\text{derivative} \quad F'"] & f = F' \arrow[l, bend left, "\text{integral}", "\int f\,dx"']
\end{tikzcd}$$

f		$F = \int f\,dx \quad (c=0)$	F' (check)
1		x	1 ✓
x		$\frac{1}{2}x^2$	$\frac{1}{2}\cdot 2x = x$ ✓
x^2		$\frac{1}{3}x^3$	$\frac{1}{3}\cdot 3x^2 = x^2$ ✓
x^n	$n \neq -1$	$\frac{1}{n+1}x^{n+1}$	$\frac{1}{n+1}(n+1)x^n = x^n$ ✓
$\frac{1}{x}$		$\ln\lvert x\rvert$	$\frac{1}{x}$ ✓

f		$F = \int f dx \quad (c=0)$	F' (check)
e^x		e^x	e^x ✓
$a^x = e^{x \cdot \ln(a)}$	$a > 0$	$\frac{1}{\ln(a)} \cdot a^x$	(*)
$\sin(x)$		$-\cos(x)$	$-[-\sin(x)] = \cos(x)$ ✓
$\cos(x)$		$\sin(x)$	$\cos(x)$ ✓
$\frac{1}{1+x^2}$		$\arctan(x)$	$\frac{1}{1+x^2}$ ✓

(*) $\left[\frac{1}{\ln(a)} a^x\right]' = \left[\frac{1}{\ln(a)} e^{x \cdot \ln(a)}\right]' = \cancel{\frac{1}{\ln(a)}} \cdot e^{x \cdot \ln(a)} \cdot \cancel{\ln(x)} = a^x$ ✓

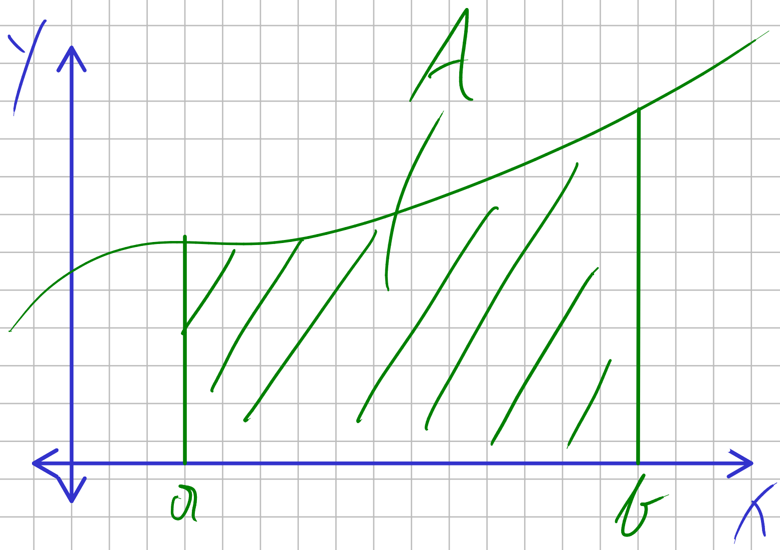
Important rules

$\int c\, f(x)\, dx$	$= c \int f(x)\, dx$	constant factor
$\int f_1(x) + f_2(x)\, dx$	$= \int f_1(x)\, dx + \int f_2(x)\, dx$	sum rule
$\int g'(x)\; f(g(x))\, dx$	$= \int f(u)\, du$	substitution
$\int f'(x) \cdot g(x)\, dx$	$= f(x) \cdot g(x) - \int f(x) \cdot g(x)\, dx$	integration by parts

Definite Integral

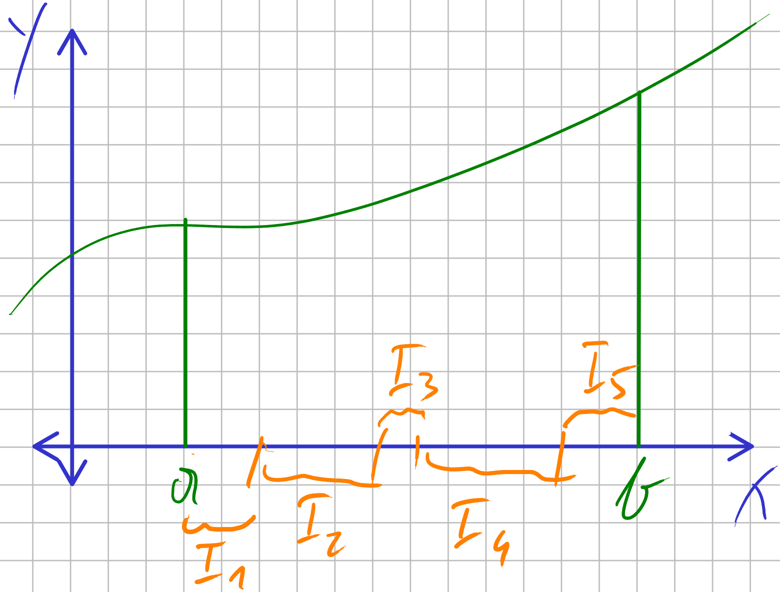
Let f be a bounded function with $f: [a,b] \to \mathbb{R}$ and $\ell \le f \le L$

Goal: compute areas e.g. for $f \ge 0$ and $G = \{(x,y) : a \le x \le b,\ 0 \le y \le f(x)\}$



Notation $A = \int_a^b f(x)\,dx$

Definite Integrals - the Riemann integral



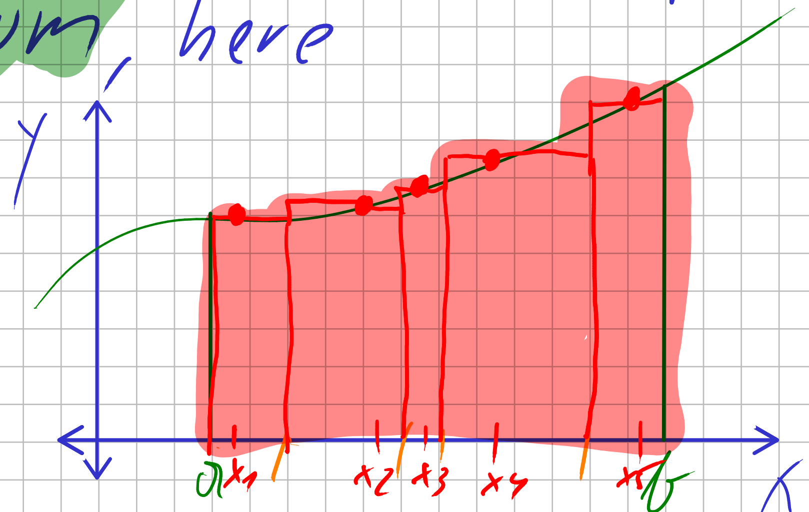
1) **Partition of interval** $[a,b]$

Length of subintervals $|I_i|$ can be differently

2) In each subinterval take an **arbitrary point** $x_i \in I_i$ and compute $f(x_i)$ and then compute the **Riemann sum**, here

$$R = \sum_{i=1}^{5} f(x_i) \cdot |I_i|$$

→ area of 5 rectangles



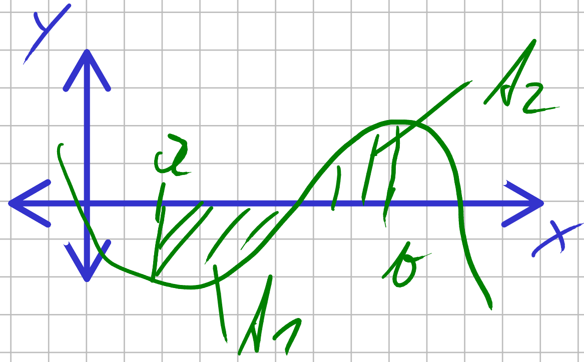
3) Partition of $[a,b]$ gets finer by using more and more subintervals such that $\max |I_i| \to 0$

In step 2 R depends on the chosen subintervals I_i and points x_i

If R converges to a value A and A is the same for all possible subintervals I_i and for all possible x_i then we say f is integrable in the sense of Riemann and we write

$$A = \int_a^b f(x)\,dx$$

Remarks:



- If f also negative, then we get signed area
  $A = A_1 + A_2$ with $A_1 < 0$ or choose pos. value $A = |A_1| + |A_2|$
- All piece wise continuous functions are integrable
  $\Rightarrow$ all continuous functions are integrable
  $\Rightarrow$ all functions with existing 1^{st} derivative are integrable
- Numerical approach for computing A:
  take midpoint of each interval and the Riemann sum R
  is an approximation of A Midpoint rule
- To compute A exactly we need ...

... the **fundamental theorem of calculus**

$$G(x) = \int_a^x f(t)\,dt$$ is an antiderivative of f i.e.

$$G'(x) = f(x)$$

Conclusion Take any antiderivative $F(x)$ of $f(x)$

$\Rightarrow F(x) = G(x) + c$ all antiderivatives are the same up to a constant c

$\Rightarrow F(a) = G(a) + c = c$ because $\int_a^a f(t)\,dt = 0$ $\Rightarrow c = F(a)$

$\Rightarrow F(b) = G(b) + c$ $\Rightarrow \int_a^b f(t)\,dt = G(b) = F(b) - c$

$$\Rightarrow \int_a^b f(t)\,dt = F(b) - C = F(b) - F(a) =: F(t)\Big|_a^b$$

We can rename t by x to get

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$

For interested readers only

Antiderivatives are equal up to a constant

Let F and G be two antiderivatives of f i.e. $F' = G' = f$

$\Rightarrow (F-G)' = F' - G' = f - f = 0$

$\Rightarrow F - G$ is constant $\Rightarrow F = G + c$

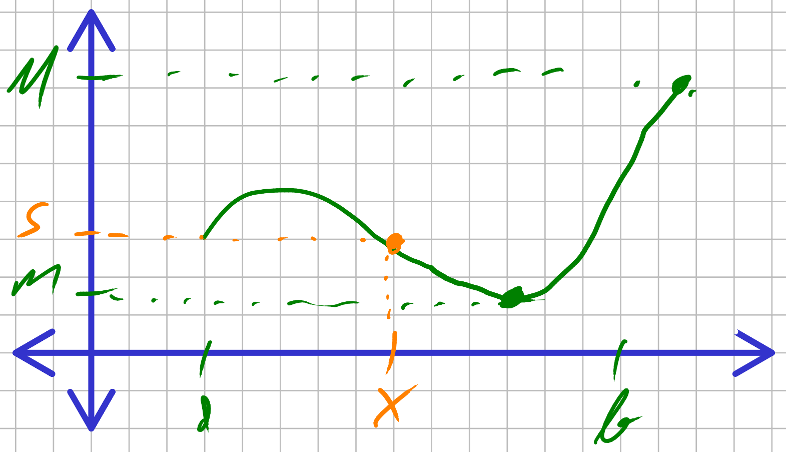
is somehow clear but has to be proven by mean value theorem of differential calculus

For interested readers only

1) Intermediate Value Theorem

$f:[a,b]\to\mathbb{R}$ continuous and $m=\min_{x\in[a,b]} f$ $\quad M=\max_{x\in[a,b]} f$

$\Rightarrow$ For any $s\in\mathbb{R}$ with $m<s<M$ there exists a $x\in[a,b]$ with $f(x)=s$



Proof ... Intuitive

For interested readers only

2) Mean Value Theorem for definite integrals

$f: [a,b] \to \mathbb{R}$ continuous and $m = \min\limits_{x\in[a,b]} f$ $\quad M = \max\limits_{x\in[a,b]} f$

$$\Rightarrow m \cdot (b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

$$\Rightarrow m \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq M$$

$\Rightarrow$ Intermediate value theorem with $s = \frac{1}{b-a} \int_a^b f(x)\,dx$

$\Rightarrow$ There exists a $x \in [a,b]$: $\quad f(x) = \frac{1}{b-a} \int_a^b f(x)\,dx$

For interested readers only

Proof of fundamental theorem of calculus

$$G(x) := \int_0^x f(t)\,dt$$

In x_0 the derivative is $G'(x_0) = \lim_{\Delta x \to 0} \frac{G(x_0+\Delta x) - G(x_0)}{\Delta x} =$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \cdot \left[\int_0^{x_0+\Delta x} f(t)\,dt - \int_0^{x_0} f(t)\,dt \right] = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{x_0}^{x_0+\Delta x} f(t)\,dt = \lim_{\Delta x \to 0} f(x) \overset{(*)}{=}$$

$\Rightarrow$ Mean value theorem: there exists a $x \in [x_0, x_0+\Delta x]$ such that

$\overset{(*)}{=} f(x_0)$ because $x = x_0$ is only element of $[x_0, x_0+\Delta x]$ if $\Delta x \to 0$ Q.E.D.

For interested readers only

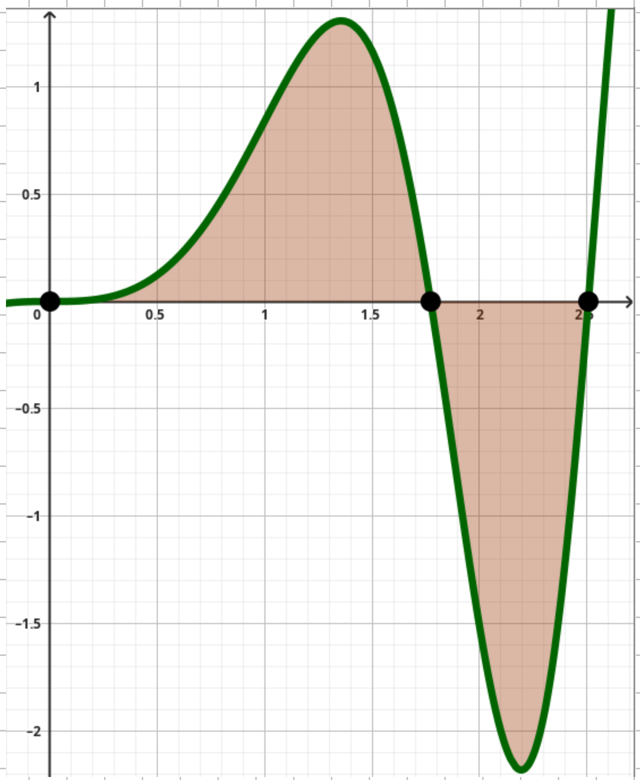
Theorem	(Satz, Theorem)	important mathematical result
Lemma	(Hilfssatz, Lemma)	An auxiliary mathematical result
Proposition	(Proposition)	A mathematical result which is not needed for a theorem and not important enough to be a theorem
Corollary	(Folgerung, Korollar)	A math. result which follows directly from a theorem, proposition or lemma
Proof	(Beweis)	The very heart of math. practice.

Proof
- → direct
- → indirect
- → proof by induction

Q.E.D. means Quod Erat Demonstrandum "which was to be demonstrated"
→ Proven

Example Compute the positive area for $f = x \cdot \sin(x^2)$ and $x \in [0, b]$ where b is the 2nd positive root of f.

Use midpoint rule to approximate the result



Roots: $x_1 = 0 \quad x_2 = \sqrt{\pi} \quad x_3 = \sqrt{2\pi}$

Antiderivative: $F(x) = -\frac{1}{2} \cdot \sin(x^2) + C$

Area: $A = 1 + |-1| = 2$

Hint: use substitution

Midpoint rule with 20 subintervals

equidistant here

absolute value

left l	right r	midpoint m = (l + r) / 2	f value f = f(m)	interval length d = r – l	area rectangle a = |f * d|
0.000	0.125	0.063	0.000	0.125	0.000
0.125	0.251	0.188	0.007	0.125	0.001
0.251	0.376	0.313	0.031	0.125	0.004
0.376	0.501	0.439	0.084	0.125	0.011
0.501	0.627	0.564	0.176	0.125	0.022
0.627	0.752	0.689	0.315	0.125	0.040
0.752	0.877	0.815	0.502	0.125	0.063
0.877	1.003	0.940	0.727	0.125	0.091
1.003	1.128	1.065	0.966	0.125	0.121
1.128	1.253	1.191	1.177	0.125	0.147
1.253	1.379	1.316	1.299	0.125	0.163
1.379	1.504	1.441	1.260	0.125	0.158
1.504	1.629	1.567	0.994	0.125	0.125
1.629	1.755	1.692	0.466	0.125	0.058
1.755	1.880	1.817	-0.291	0.125	0.037
1.880	2.005	1.943	-1.148	0.125	0.144
2.005	2.131	2.068	-1.875	0.125	0.235
2.131	2.256	2.193	-2.183	0.125	0.274
2.256	2.381	2.319	-1.826	0.125	0.229
2.381	2.507	2.444	-0.746	0.125	0.094
					2.014 $\approx A$

some negative values

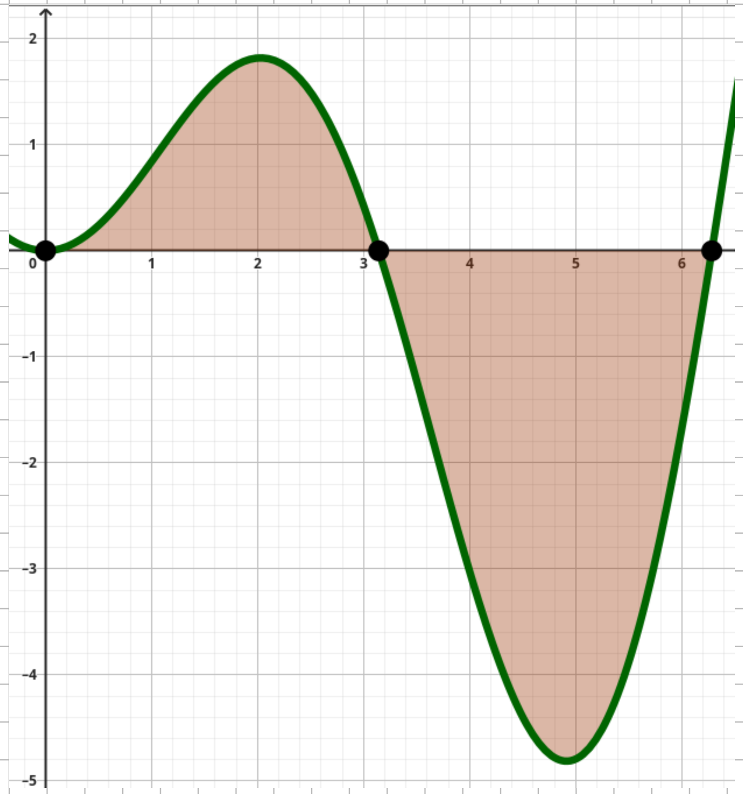
Example | same as above with $f = x \cdot \sin(x)$

Roots: $x_1 = 0 \quad x_2 = \pi \quad x_3 = 2\pi$

Antiderivative: $F(x) = -x \cdot \cos(x) + \sin(x) + c$

Area: $A = \pi + |-3\pi| = 4\pi \approx 12.57$

Hint: use integration by parts



Midpoint rule with 20 subintervals

equidistant here

absolute value

left l	right r	midpoint m = (l + r) / 2	f value f = f(m)	interval length d = r – l	area rectangle a = |f * d|
0.000	0.314	0.157	0.025	0.314	0.008
0.314	0.628	0.471	0.214	0.314	0.067
0.628	0.942	0.785	0.555	0.314	0.174
0.942	1.257	1.100	0.980	0.314	0.308
1.257	1.571	1.414	1.396	0.314	0.439
1.571	1.885	1.728	1.707	0.314	0.536
1.885	2.199	2.042	1.819	0.314	0.572
2.199	2.513	2.356	1.666	0.314	0.523
2.513	2.827	2.670	1.212	0.314	0.381
2.827	3.142	2.985	0.467	0.314	0.147
3.142	3.456	3.299	-0.516	0.314	0.162
3.456	3.770	3.613	-1.640	0.314	0.515
3.770	4.084	3.927	-2.777	0.314	0.872
4.084	4.398	4.241	-3.779	0.314	1.187
4.398	4.712	4.555	-4.499	0.314	1.413
4.712	5.027	4.869	-4.810	0.314	1.511
5.027	5.341	5.184	-4.619	0.314	1.451
5.341	5.655	5.498	-3.888	0.314	1.221
5.655	5.969	5.812	-2.639	0.314	0.829
5.969	6.283	6.126	-0.958	0.314	0.301
					12.618

$\approx A$

some negative values

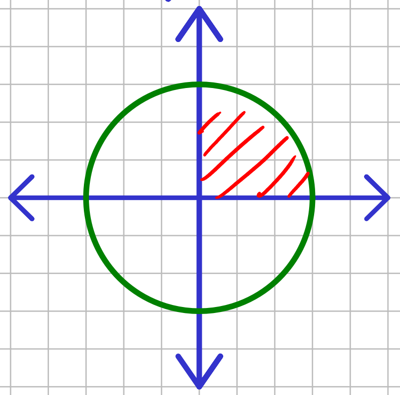
Example Find the antiderivative of $\cos^2(t)$

Hint: Use integration by parts and then $\sin^2(t) + \cos^2(t) = 1$

$$F(t) = \frac{t}{2} + \frac{\sin(t)\cos(t)}{2} + c$$

$\rightarrow$ See exercises

Example: Compute area of a circle with radius R



Hint 1: compute red area $\tilde{A}$ $\Rightarrow A = 4 \cdot \tilde{A}$

$\Rightarrow y = \sqrt{R^2 - x^2} \quad x \in [0, R]$

Hint 2: use substitution $x = R \cdot \sin(t) \quad t \in [0, \frac{\pi}{2}]$

[$x = R\cos(t)$ possible too but signs worse]

be careful here

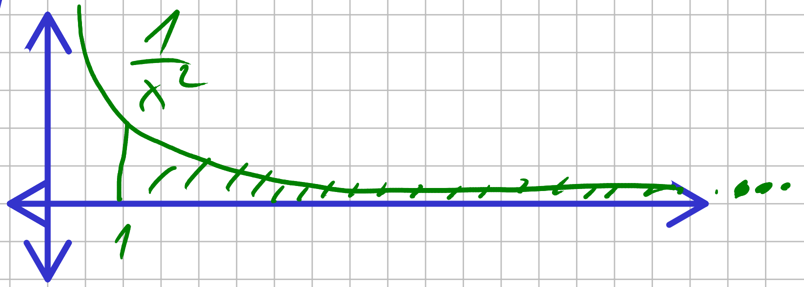
$\Rightarrow R^2 - x^2 = \ldots\ R^2 \cos^2(t) \quad \Rightarrow \sqrt{R^2 - x^2} = |R\cos(t)| = R\cos(t)$

$\frac{dx}{dt} = R\cos(t) \quad \Rightarrow \quad dx = R\cos(t)\,dt$

$\Rightarrow \tilde{A} = R^2 \int_0^{\frac{\pi}{2}} \cos^2(t)\,dt = R^2 \left[\frac{t}{2} + \frac{\sin(t)\cos(t)}{2} \right]_0^{\frac{\pi}{2}} = \frac{R^2 \pi}{4} \quad \Rightarrow A = R^2 \pi$

Indefinite Integral

Till now interval $[a,b]$ bounded and f bounded



If e.g. $b=\infty$ and f bounded we define

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty}\int_a^R f(x)\,dx$$

And if $a=-\infty$ we define $\int_{-\infty}^{b} f(x)\,dx = \lim_{S\to\infty}\int_{-S}^{b} f(x)\,dx$

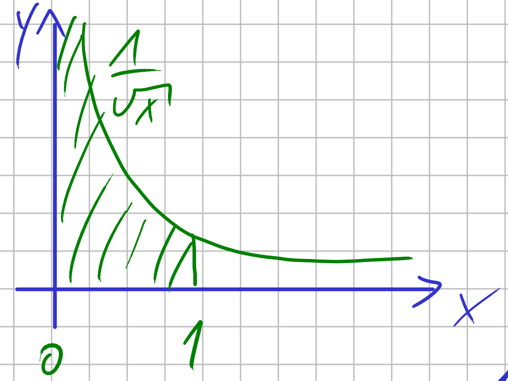
And if $a=-\infty$ and $b=\infty$ $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$

E.g. use 0 or any other real number

Indefinite Integral

If $[a,b]$ is bounded but f unbounded in $[a,b]$

E.g. $f(b)=\infty$ we define



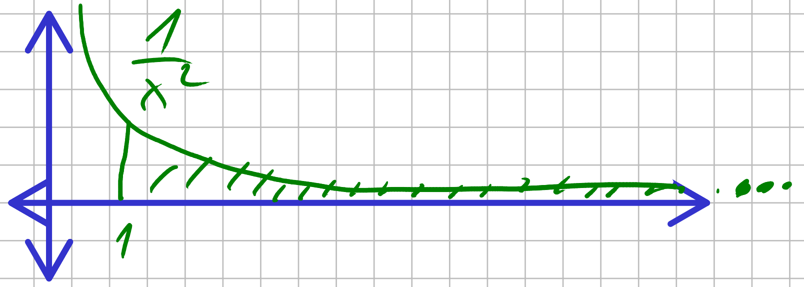
$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)\,dx$$

And problem is in point a we define $\int_a^b f(x)\,dx = \lim_{\delta \to 0} \int_{a+\delta}^b f(x)\,dx$

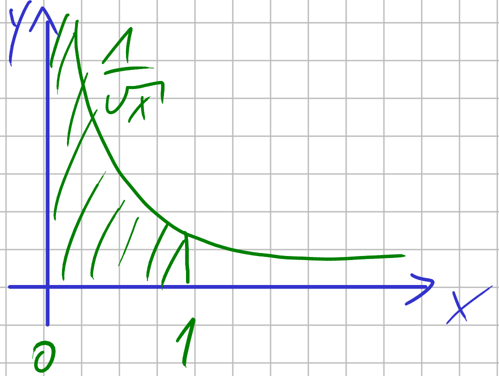
And if problem is inside in point $c \in (a,b)$: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Example! Compute

$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$

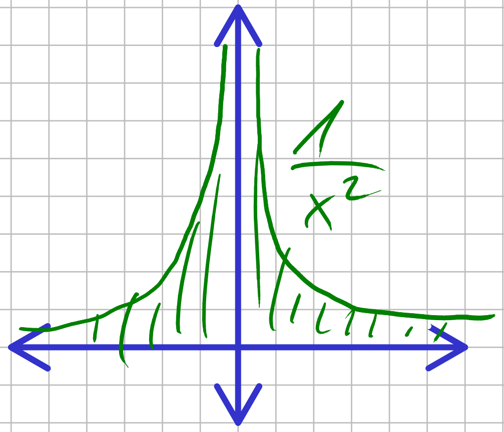


$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$$



$$\int_{-1}^1 \frac{1}{x^2}\,dx = \infty$$

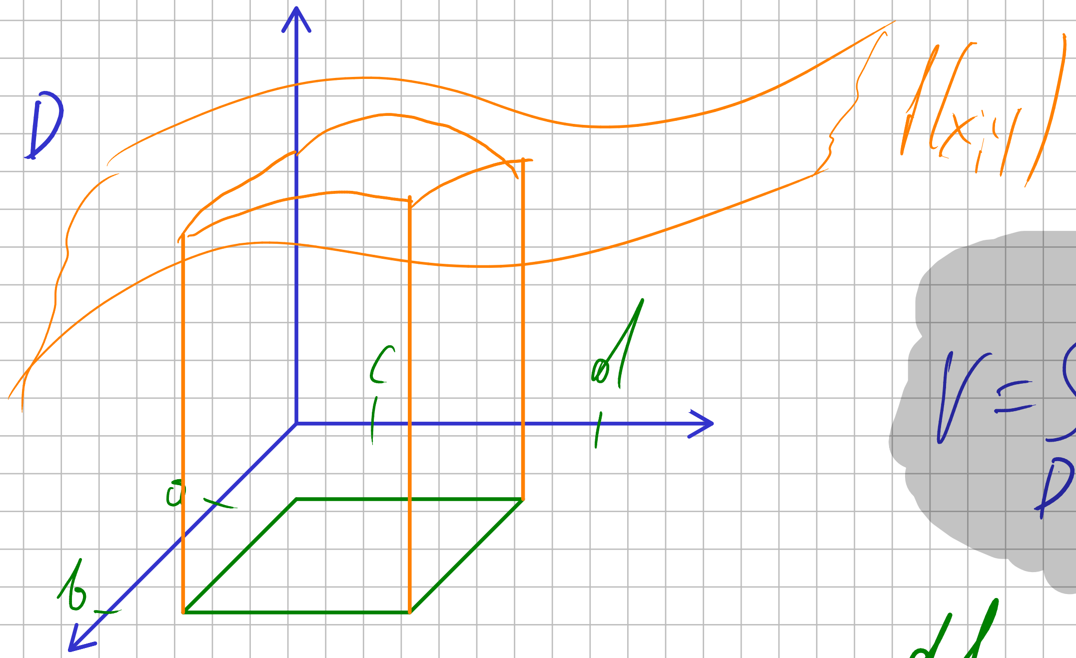
Why not $-\frac{1}{x}\Big|_{-1}^{1} = -2$



Integral in 2 or more dimensions

First approach: volumina of solid with rectangular base area and bounded function $f(x;y)$

$a \leq x \leq b$
$c \leq y \leq d$ } rectangle D
$0 \leq z \leq f(x;y)$

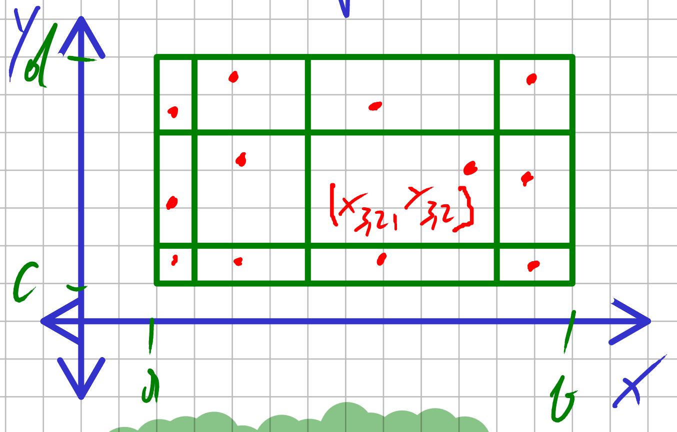


$$V = \int_D f \, dA = \iint_D f \, dA$$

dA... Area

Idea is the same as in 1d:

1) Partition of $[a,b]$ and partition of $[c,d]$



here we get $3 \cdot 4 = 12$ rectangles I_{ij}

Choose one arbitrary point (x_{ij}, y_{ij}) for each I_{ij}

Compute $f(x_{ij}, y_{ij})$

2) Riemann sum $\quad R = \sum_{i=1}^{4} \sum_{j=1}^{3} f(x_{ij}, y_{ij}) \cdot |I_{ij}|$

where $|I_i|$ is area of rectangle

3) Partition of $[a,b]$ and $[c,d]$ finer and finer such that $|I_{ij}| \to 0$

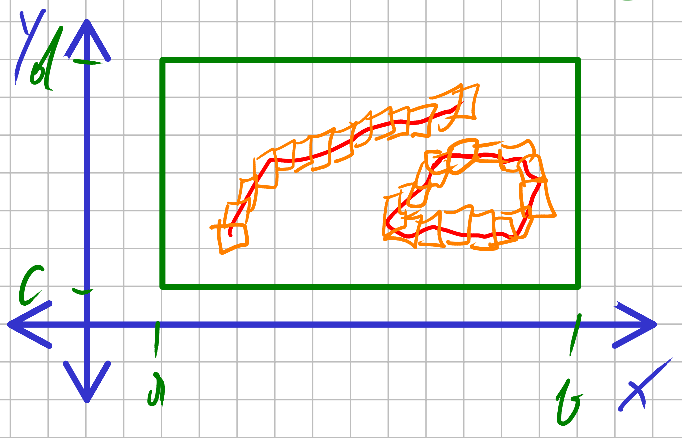
If Riemann sum converges to V and V does not depend on partition and choice of (x_{ij}, y_{ij}). then we say f is integrable in the sense of Riemann and we write

$$V = \iint_D f \, dA$$

Remarks:

- Signed volume if f is negative
- f bounded and continuous almost everywhere (discontinuous on a set of points with area 0)

$\Rightarrow f$ is Riemann integrable

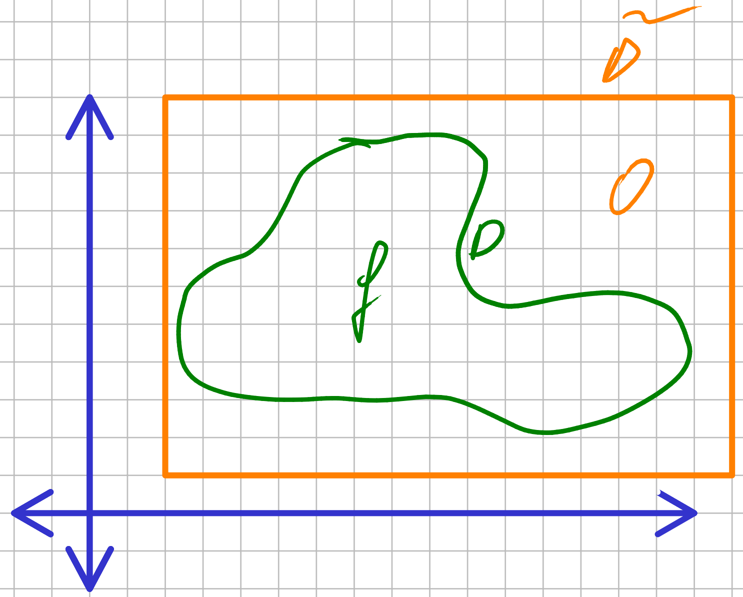


Set with area/measure 0 means: for every small $\varepsilon > 0$ the set can be covered by finitely (or countable) many rectangles such that the sum of their areas is smaller than ε.

Most curves have measure 0

- If D is not a rectangle $\Rightarrow$ large rectangle $\tilde{D}$ and integrate $\tilde{f}(x,y) = \begin{cases} f & (x,y) \in D \\ 0 & \text{else} \end{cases}$

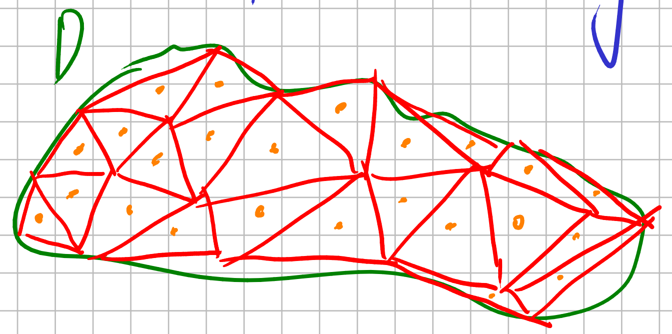
$\tilde{f}$ ... zero extension



- Midpoint rule for an approximation

use midpoint of each rectangle

or better (not here) use triangles



Example $D = [0,1] \times [0,1] = \{(x,y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$

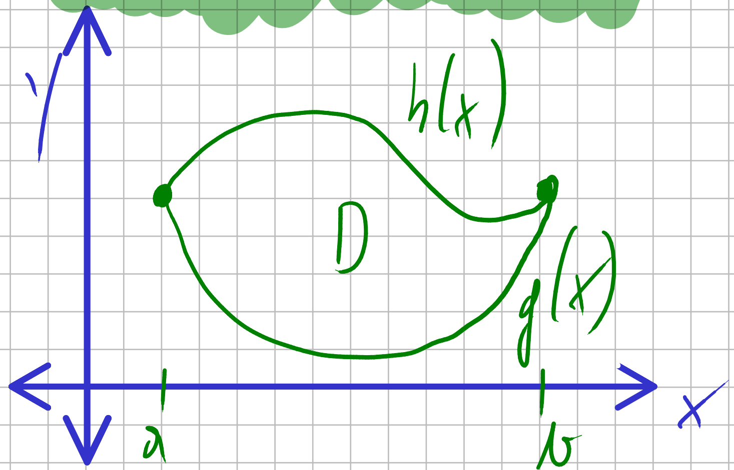
Approximate $\int_D f \, dA$ where $f = \sin(x \cdot y) + x^2 y$

x left: l	x right: r	x-midpoint: m	y left: l	y right: r	y-midpoint: n	f(m, n)	area rectangle: a	volume: a = |f * a|
0.000	0.200	0.100	0.000	0.200	0.100	0.011	0.040	0.000
0.000	0.200	0.100	0.200	0.400	0.300	0.033	0.040	0.001
0.000	0.200	0.100	0.400	0.600	0.500	0.055	0.040	0.002
0.000	0.200	0.100	0.600	0.800	0.700	0.077	0.040	0.003
0.000	0.200	0.100	0.800	1.000	0.900	0.099	0.040	0.004
0.200	0.400	0.300	0.000	0.200	0.100	0.039	0.040	0.002
0.200	0.400	0.300	0.200	0.400	0.300	0.117	0.040	0.005
0.200	0.400	0.300	0.400	0.600	0.500	0.194	0.040	0.008
0.200	0.400	0.300	0.600	0.800	0.700	0.271	0.040	0.011
0.200	0.400	0.300	0.800	1.000	0.900	0.348	0.040	0.014
0.400	0.600	0.500	0.000	0.200	0.100	0.075	0.040	0.003
0.400	0.600	0.500	0.200	0.400	0.300	0.224	0.040	0.009
0.400	0.600	0.500	0.400	0.600	0.500	0.372	0.040	0.015
0.400	0.600	0.500	0.600	0.800	0.700	0.518	0.040	0.021
0.400	0.600	0.500	0.800	1.000	0.900	0.660	0.040	0.026
0.600	0.800	0.700	0.000	0.200	0.100	0.119	0.040	0.005
0.600	0.800	0.700	0.200	0.400	0.300	0.355	0.040	0.014
0.600	0.800	0.700	0.400	0.600	0.500	0.588	0.040	0.024
0.600	0.800	0.700	0.600	0.800	0.700	0.814	0.040	0.033
0.600	0.800	0.700	0.800	1.000	0.900	1.030	0.040	0.041
0.800	1.000	0.900	0.000	0.200	0.100	0.171	0.040	0.007
0.800	1.000	0.900	0.200	0.400	0.300	0.510	0.040	0.020
0.800	1.000	0.900	0.400	0.600	0.500	0.840	0.040	0.034
0.800	1.000	0.900	0.600	0.800	0.700	1.156	0.040	0.046
0.800	1.000	0.900	0.800	1.000	0.900	1.453	0.040	0.058
								0.405

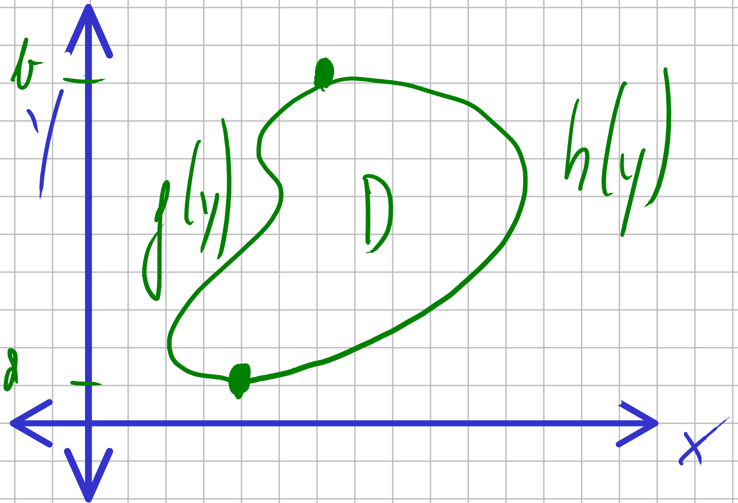
exact would be 0.40647...

Open problem: how to compute integral analytically (not numerically)

Fubini's Theorem



$$\int_D f\,dA = \overbrace{\int_{x=a}^{b} \underbrace{\int_{y=g(x)}^{h(x)} f(x,y)\,dy}_{\text{1d integral in } y}\,dx}^{\text{1d integral in } x}$$



$$\int_D f\,dA = \overbrace{\int_{y=a}^{b} \underbrace{\int_{x=g(y)}^{h(y)} f(x,y)\,dx}_{\text{1d integral in } x}\,dy}^{\text{1d integral in } y}$$

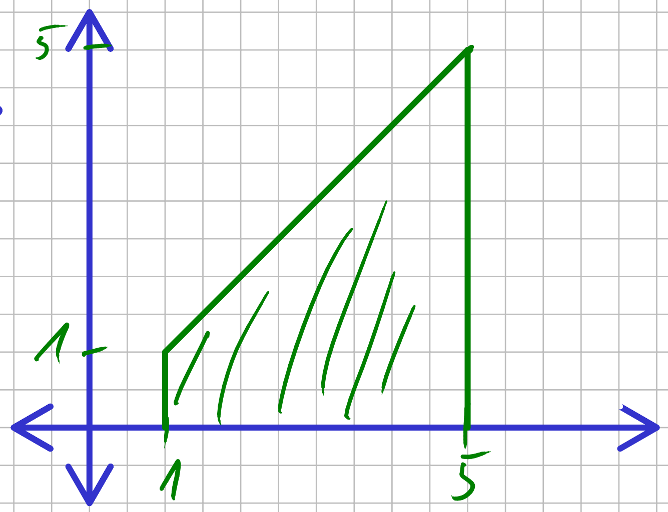
Example!

Compute $\int_D f \, dA$ with $D = [1,5] \times [0,4]$, $f = 3x^2 y$

Solution: 992

Example!

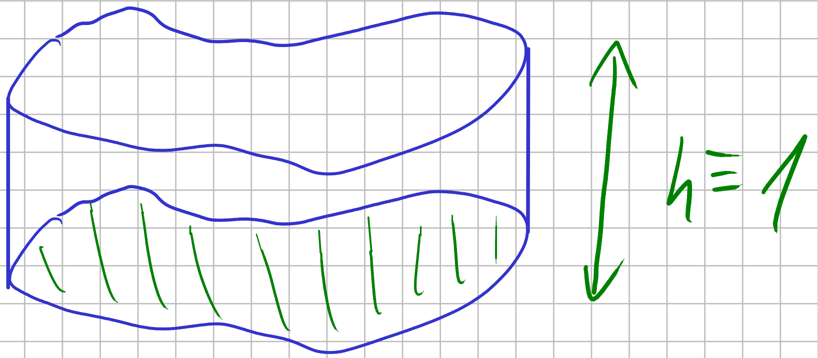
Compute $\int_D f \, dA$ with $f = 3x^2 y$ and D:



Solution: 937,2

With double integral we can compute volumes but also areas!, just use $f \equiv 1$ (height)

$\Rightarrow A = V$



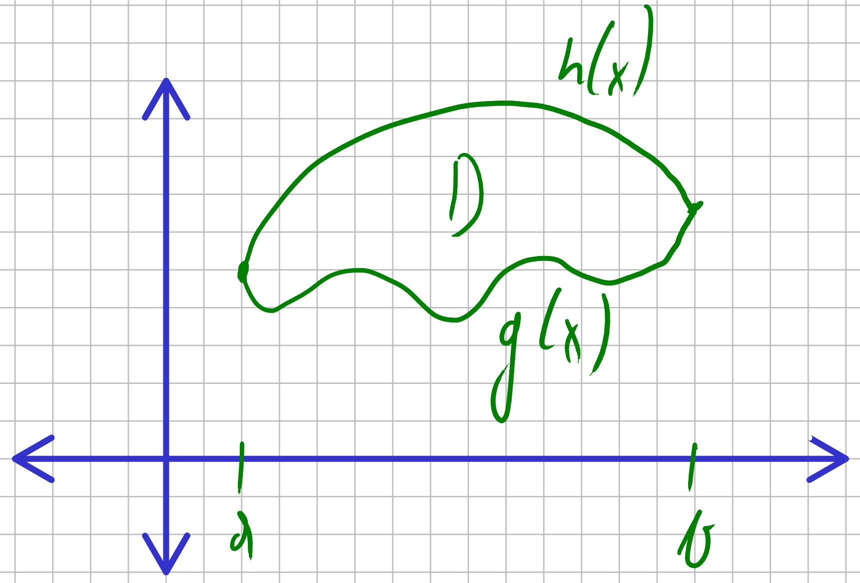
$A = ?$

$V = A \cdot 1 = A$

$$\Rightarrow A = V = \int_D 1 \, dA$$

But we knew this already because:

If $D = \{(x,y) : a \leq x \leq b$ and $g(x) \leq y \leq h(x)\}$



$$A = |D| = \int_D 1 \, dA = \int_{x=a}^{b} \int_{y=g(x)}^{h(x)} 1 \, dy \, dx =$$

$$= \int_{x=a}^{b} y \Big|_{g(x)}^{h(x)} dx = \int_{x=a}^{b} h(x) - g(x) \, dx$$

This we know from school / discrete math / calculus

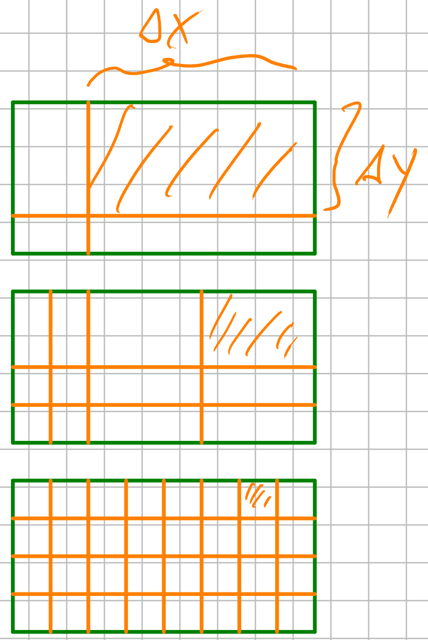
Cartesian Coordinates (named after René Descartes 1596-1650)

Previous idea

$$\Delta A = \Delta x \cdot \Delta y \quad \text{or} \quad \Delta A = \Delta y \cdot \Delta x$$

$\downarrow$ finer and finer mesh

$$dA = dx\,dy \quad \text{oder} \quad dA = dy\,dx$$

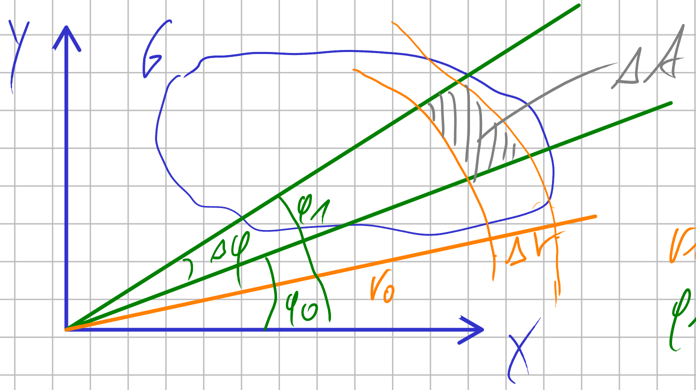


$\vdots$

Polar Coordinates instead of (x,y) use (r,φ) radius and angle

~~$dA = d\varphi\, dr$ is wrong!~~

$dA = r\, dr\, d\varphi$ or $dA = r\, d\varphi\, dr$



$r_1 = r_0 + \Delta r$

$\varphi_1 = \varphi_0 + \Delta\varphi$

$$\Delta A = \frac{(r_0+\Delta r)^2 \pi \Delta\varphi}{2\pi} - \frac{r_0^2 \pi \Delta\varphi}{2\pi}$$

$$= \frac{(r_0^2 + 2 r_0 \Delta r + \Delta r^2 - r_0^2)\,\Delta\varphi}{2}$$

$$= r_0 \Delta r \Delta\varphi + \underbrace{\frac{1}{2}\Delta r^2 \Delta\varphi}_{\text{small}} \approx r_0 \Delta r \Delta\varphi$$

Polar Coordinates

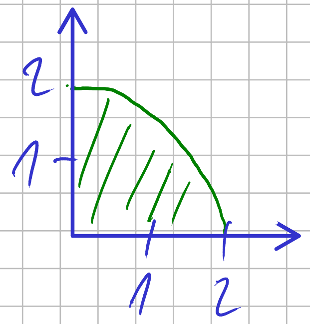
$$\int_G f\,dA = \int_{r=a}^{b} \int_{\varphi=c}^{d} f(r,\varphi)\, r\, d\varphi\, dr$$

or if φ depends on r: $$\int_G f\,dA = \int_{r=a}^{b} \int_{\varphi=g(r)}^{h(r)} f(r,\varphi)\, r\, d\varphi\, dr$$

or if r depends on φ: $$\int_G f\,dA = \int_{\varphi=a}^{b} \int_{r=g(\varphi)}^{h(\varphi)} f(r,\varphi)\, r\, dr\, d\varphi$$

Example $f = x^2 + y^2 \Rightarrow f = r^2$

Domain D:



a) Cartesian: $V = \int_{x=0}^{2} \int_{y=0}^{\sqrt{4-x^2}} x^2 + y^2 \, dy \, dx = \ldots = 6{,}283\ldots$

Python

b) Polar Coordinates: $V = \int_{r=0}^{2} \int_{\varphi=0}^{\frac{\pi}{2}} r^2 \cdot r \, d\varphi \, dr = \ldots = 2\pi = 6{,}283\ldots$