Exercise Sheet 7: Gradient Descent and Lagrange Multiplier

Example 1 *Minimize the function*

$$f(x;y) = 5 + 5x^2 + 2y^2 - 10x - 6y + 6xy$$

by using the gradient descent method. Use $x_0 = 1$ and $y_0 = 1$ as initial solution. Compute 3 iterations with step size 0.1.

Solution, see python and figure 1.

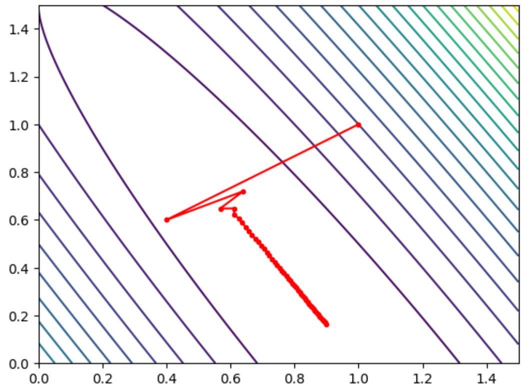

Figure 1: Path of gradient descent method

Check:
$f_x = 10x - 10 + 6y = 0$
$f_y = 4y - 6 + 6x = 0$
$x = 1 - 0.6y$ *(1^{st} equation) and therefore (2^{nd} equation)* $4y - 6 + 6 - 3.6y = 0.4y = 0$, *hence* $y = 0$ *and* $x = 1$.

Example 2 *A company wants to produce* 50 *units of a special product as cheap as possible. Using K units* capital *and L units* labour, $\sqrt{K} + L$ *units can be manufactured. The costs for capital and labour are* 1 *and* 20 *respectively. Find the optimal values for K und L:*

(a) Use Lagrangian's method.

(b) Sketch the feasible set.

(c) Are there singular points, i.e. points with $\nabla\varphi = \vec{0}$?

(d) Consider the boundaries of the feasible set and find the global minimum.

Solution:
Minimize $f(K,L) = K + 20L$ *s.t.* $\varphi(K,L) = \sqrt{K} + L - 50 = 0$,

$$\begin{aligned}
\mathcal{L}(K,L,\lambda) &= K + 20L + \lambda \cdot \left(\sqrt{K} + L - 50\right) \\
\frac{\partial \mathcal{L}}{\partial K}(K,L,\lambda) &= 1 + \lambda \frac{1}{2} K^{-\frac{1}{2}} = 1 + \frac{\lambda}{2\sqrt{K}} = 0 \\
\frac{\partial \mathcal{L}}{\partial L}(K,L,\lambda) &= 20 + \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda}(K,L,\lambda) &= \sqrt{K} + L - 50 = 0.
\end{aligned}$$

$\lambda = -20$, $K = 100$, $L = 40$ *is minimum because* $f(100,40) = 900$ *and boundaries* $f(0,50) = 1000 > 900$ *and* $f(2500,0) = 2500 > 900$. *There are no singular points. Feasible set, see figure 2.*

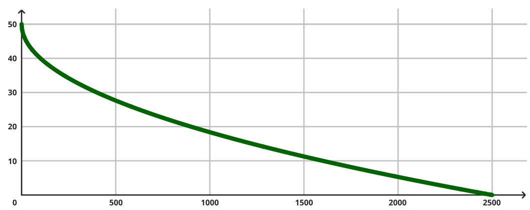

Figure 2: Feasible set

Example 3 *Somebody has the utility function* $u(x,y) = 100xy + x + 2y$. *The price of the first good* x *is* 2 *Euro and for the second good* y *it is* 4 *Euro. The person owns* 1000 *Euro, which have to be spent! Find the solution which maximizes the utility function and sketch the feasible set.*

Solution:
Maximize $f(x,y) = 100xy + x + 2y$ *s.t.* $\varphi(x,y) = 2x + 4y - 1000 = 0$,

$$\begin{aligned}
\mathcal{L}(x,y,\lambda) &= 100xy + x + 2y + \lambda \cdot (2x + 4y - 1000) \\
\frac{\partial \mathcal{L}}{\partial x}(x,y,\lambda) &= 100y + 1 + 2\lambda \overset{!}{=} 0 \\
\frac{\partial \mathcal{L}}{\partial y}(x,y,\lambda) &= 100x + 2 + 4\lambda \overset{!}{=} 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda}(x,y,\lambda) &= 2x + 4y - 1000 \overset{!}{=} 0
\end{aligned}$$

We get the global maximum at $x = 250$ *and* $y = 125$ *with* $f(250,125) = 3\,125\,500$, *because on the boundary we get* $f(500,0) = 500$ *and* $f(0,250) = 500$. *There are no singular points, i.e. points with* $\nabla\varphi = \vec{0}$. *Feasible set, see figure 3.*

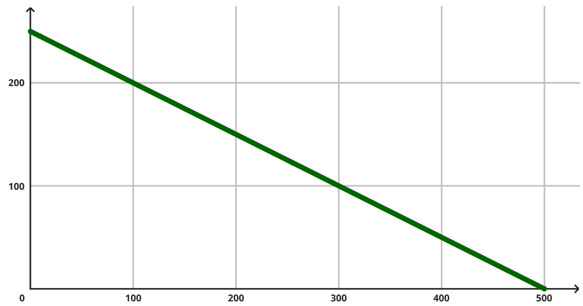

Figure 3: Feasible set

In case you want to practice further optimization problems, look at the following. You can use Lagrange's method or a parametrization of the boundary.

Example 4 *Find the global maximum and minimum of $f(x,y) = (x-1)\cdot(y-2)$ in the domain*

$$D_1 := \left\{(x,y) \colon x \geq 0, y \geq 0 \text{ and } y \leq 4 - \frac{x}{2}\right\}.$$

Sketch the domain before optimizing.

Solution:
In figure 4 you can find all critical points. Critical points see table 1.

	x	y	z
P_1	1	2	0
P_2	0	0	2
P_3	8	0	-14
P_4	0	4	-2
P_5	2.5	2.75	1.125

Table 1: Solution for D_1

Global minimum at P_3 and global maximum at P_2.

Example 5 *Find the global maximum and minimum of $f(x,y) = (x-1)\cdot(y-2)$ in the domain*

$$D_2 := \left\{(x,y) \colon (x-1)^2 + (y-2)^2 \leq 4\right\}.$$

Sketch the domain before optimizing. You can use Lagrange's method to solve the optimization problem on the boundary. Alternatively, you can use the parametrization of the boundary curve $\vec{\gamma}(t) = \begin{pmatrix} 1 + 2\cdot\cos t \\ 2 + 2\cdot\sin t \end{pmatrix}$, $t \in (0, 2\pi)$.

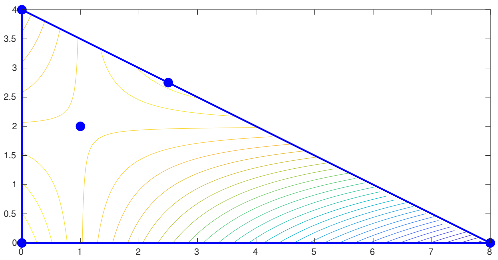

Figure 4: Contour Plot D_1

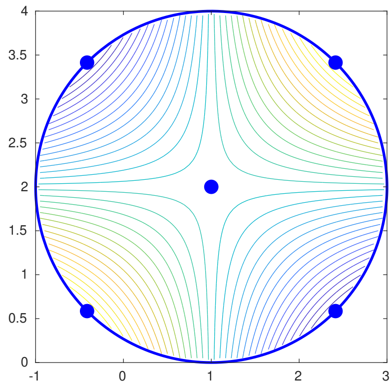

Figure 5: Contour Plot D_2

Solution:
In figure 5 you can find all critical values. Critical values see table 2. Global minimum at P_3 and P_5 and global maximum at P_2 and P_4.

	x	y	z
P_1	1	2	0
P_2	$2.414\ldots$	$3.414\ldots$	2
P_3	$-0.414\ldots$	$3.414\ldots$	-2
P_4	$-0.414\ldots$	$0.585\ldots$	2
P_5	$2.414\ldots$	$0.585\ldots$	-2

Table 2: Solution for D_2

Example 6 *Find the global maximum and minimum of $f(x,y) = x + y$ in the domain*

$$D := \left\{ (x,y)\colon y \geq 0 \text{ and } \left(x - \frac{1}{2}\right)^2 + y^2 \leq \frac{1}{4} \right\}.$$

Sketch the domain D before optimizing. To find the optimum on the boundary you can use Lagrange's method or simply use the parametrization of the boundary curve $\vec{\gamma}(t) = \begin{pmatrix} \frac{1}{2} + \frac{1}{2} \cdot \cos t \\ \frac{1}{2} \cdot \sin t \end{pmatrix}$, $t \in (0, \pi)$.

Solution:
Critical points, see table 3. Global minimum at P_1 and global maximum at P_3. See figure 6.

	x	y	z
P_1	0	0	0
P_2	1	0	1
P_3	$0.853\ldots$	$0.353\ldots$	$1.207\ldots$

Table 3: Solution for D

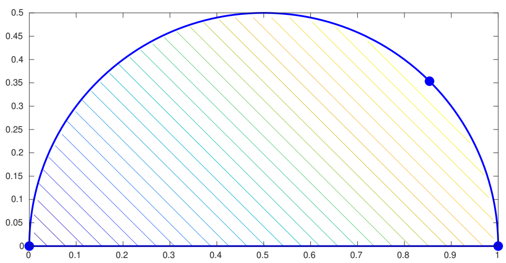

Figure 6: Contour Plot