Optimization with Constraints

Maximize / Minimize function subject to a constraint

$$\min/\max \; f(x,y)$$
$$\text{s.t. } \varphi(x,y) = 0$$

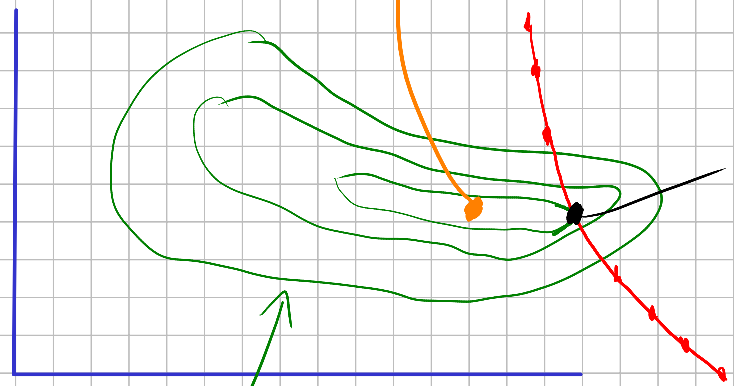
In most cases we use one constraint, but more are possible, e.g.

$$\varphi_1(x,y) = 0$$
$$\varphi_2(x,y) = 0$$
$$\vdots$$

Constraint is given by an implicit curve $\varphi(x,y) = 0$

$\longrightarrow$ only points on curve are allowed

No path on top of the mountain



On the red path this is the point with highest see level

constraint $\phi=0$
e.g. path

contour of f
e.g. see level

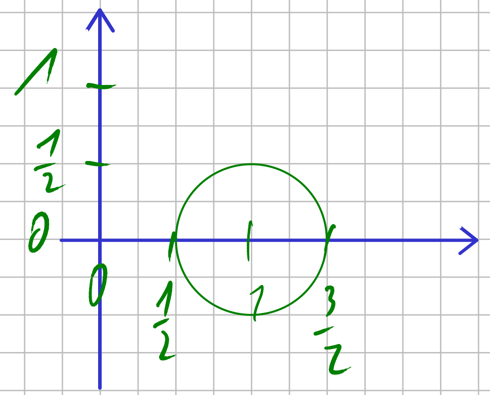
Example

Find minimum and maximum of $f(x,y) = e^{-(x^2+y^2)}$

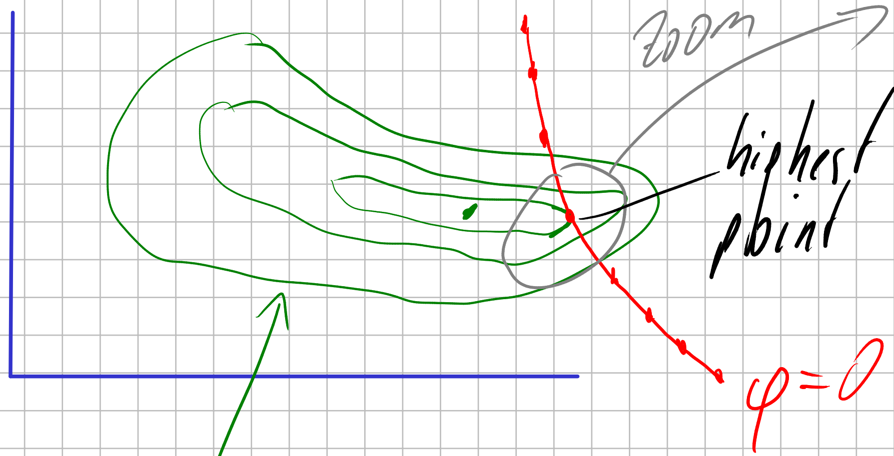
s.t. $(x-1)^2 + y^2 = \frac{1}{4}$

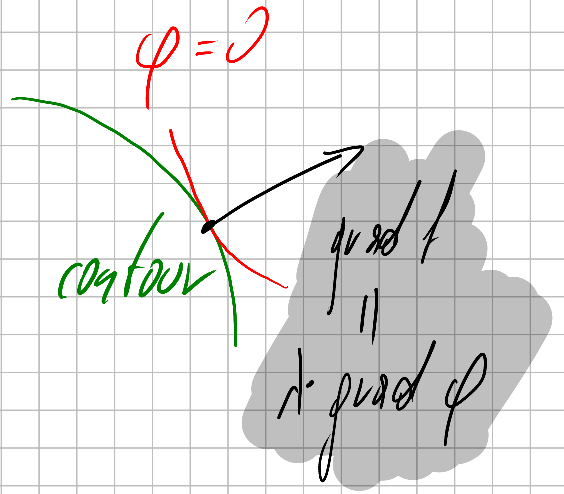
$\Leftrightarrow \underbrace{(x-1)^2 + y^2 - \frac{1}{4}}_{\varphi(x,y)} = 0$

Circle with centre $(1,0)$ and radius $\sqrt{\frac{1}{4}} = \frac{1}{2}$



Idea: Consider hiking map with sea levels as contours

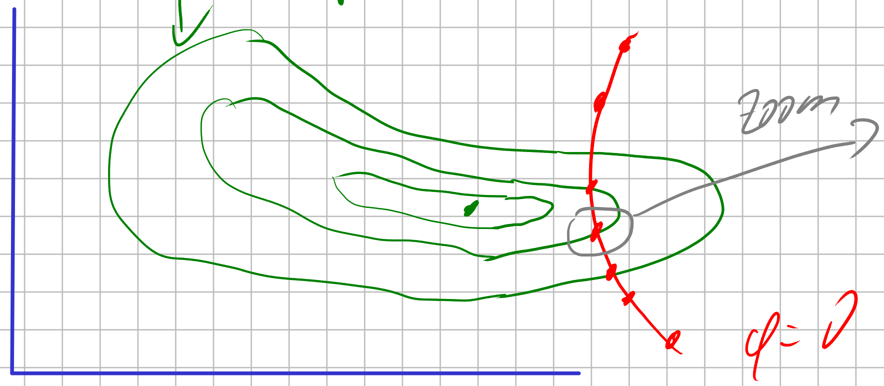


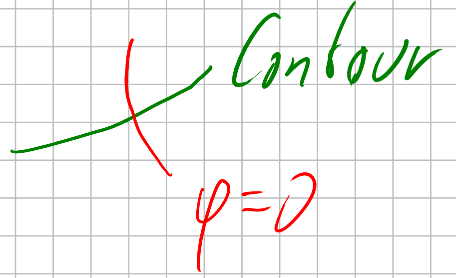


$\Rightarrow$ Both gradients show in same or opposite direction

$\Rightarrow$ This is a candidate for an optimum

Contour of f





$\Rightarrow$ path $\varphi = 0$ crosses contour

$\Rightarrow$ No optimum

Conclusion: We are interested in points (x, y) where

$$\operatorname{grad} f(x,y) = \lambda \operatorname{grad} \varphi(x,y)$$

$$\Longleftrightarrow$$

$$f_x(x,y) - \lambda \varphi_x(x,y) = 0$$
$$f_y(x,y) - \lambda \varphi_y(x,y) = 0$$

But we are not allowed to forget $\varphi(x,y) = 0 \Longleftrightarrow -\varphi(x,y) = 0$

Auxiliary function: Lagrangian Function or Lagrangian

$$\mathcal{L}(x,y,\lambda) := f(x,y) - \lambda\,\varphi(x,y)$$

$$\Rightarrow \mathcal{L}_x(x,y,\lambda) = f_x(x,y) - \lambda\,\varphi_x(x,y) \overset{!}{=} 0$$ ⟶ 1st equation

$$\mathcal{L}_y(x,y,\lambda) = f_y(x,y) - \lambda\,\varphi_y(x,y) \overset{!}{=} 0$$ ⟶ 2nd equation

$$\mathcal{L}_\lambda(x,y,\lambda) = -\varphi(x,y) \overset{!}{=} 0$$ ⟶ 3rd equation

λ is called Lagrange Multiplier

Sign is not important, we can also use $\mathcal{L} = f + \lambda\varphi$

Problematic Cases: Our idea $\operatorname{grad} f = \lambda \operatorname{grad} \varphi$
does not always work

Case 1: What happens if $\operatorname{grad} f = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

We get this critical point anyway because if $\lambda = 0$ it solves

$$\underbrace{\operatorname{grad} f}_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} = \underbrace{\lambda}_{0} \operatorname{grad} \varphi$$

independently on φ $\rightarrow$ no problem

Case 2: What if $\operatorname{grad} f \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\operatorname{grad} \varphi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\rightarrow$ singular points of $\varphi = 0$

$$\underbrace{\operatorname{grad} f}_{\neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}} = \lambda \underbrace{\operatorname{grad} \varphi}_{= \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \quad \text{↯}$$

$\Rightarrow$ We have to handle singular points separately

Example) Find min/max of $1-x^2-y^2$
s.t. $2x-y+1=0$

global min for $(x,y) \to (\infty,\infty)$ $z=-\infty$
or $(x,y) \to (-\infty,-\infty)$ $z=-\infty$

global max at $(-0.4, 0.2)$ $z=0.8$

Example) Find min/max of $\exp(-x^2-y^2)$
s.t. $(x-1)^2+y^2=\frac{1}{4}$

global min at $(0.5, 0)$ $z \approx 0.78$
global max at $(1.5, 0)$ $z \approx 0.11$

Example/ Find min/max of $\exp(-x-y)$

s.t. $x \cdot y = 0$

Singular point $(0,0)$ $z = 1$

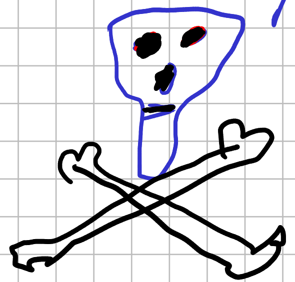
Global max at $(-\infty, 0)$ and $(0, -\infty)$ $z = \infty$

Global min at $(\infty, 0)$ and $(0, \infty)$ $z = 0$

Remark

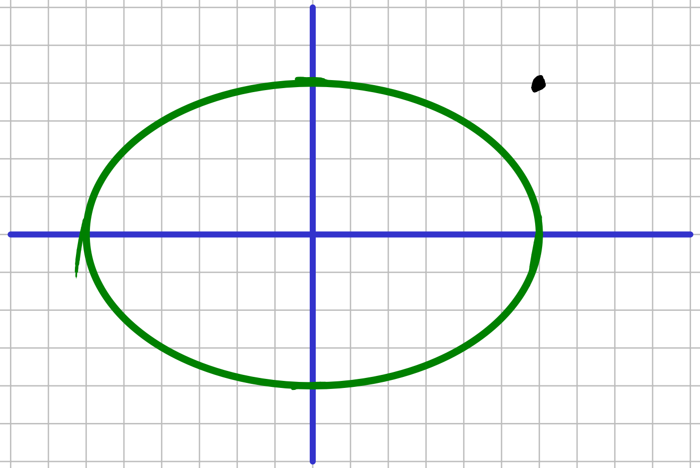
- Most examples are trivial $\rightarrow$ 1d problems possible

  See eg examples from above $\rightarrow$ For test it is forbidden to simplify



- Real examples are too complicated without computer
- Use Newton's method to solve nonlinear system

Example Find min/max of distance (x,y) to $(3,2)$ where
$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ ellipse



$$\Rightarrow f(x,y) = (x-3)^2 + (y-2)^2$$
$$\text{s.t.} \quad \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

Min at $(2.36,\ 1.24)$ $\quad \sqrt{f} \approx 1.00$

Max at $(-2.88,\ -0.56)$ $\quad \sqrt{f} \approx 6.41$

Optimization and Domain

<u>Example</u> Find min/max of $2x+5y$

$(x,y) \in \{(x,y): x^2+y^2 \leq 4\}$

a) Solve with parametric curve $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2\cos(t) \\ 2\sin(t) \end{pmatrix}$ $t \in [0, 2\pi)$

b) Use Lagrange Multiplier $x^2+y^2=4$

global max at $(0.74, 1.86)$ $z = 10.77$ min at $(-0.74, -1.86)$ $z = -10.77$