Functions and Curves

1d function (known)

$$f: D \subseteq \mathbb{R} \longrightarrow R \subseteq \mathbb{R}$$
$$x \longmapsto f(x)$$

e.g. $D = R = \mathbb{R}$ $\quad f(x) = x^2 + 2x + 1$

2d function (known)

$$f: D \subseteq \mathbb{R}^2 \longrightarrow R \subseteq \mathbb{R}$$
$$(x,y) \longmapsto f(x,y)$$

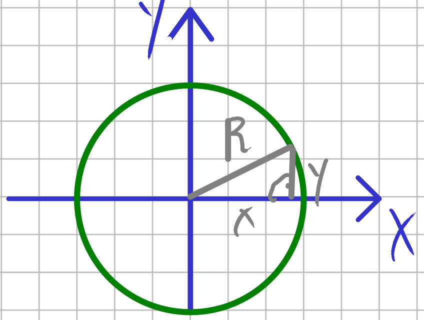
e.g. $D = \mathbb{R}^2$ $\quad R = \mathbb{R}$ $\quad f(x,y) = 5xy - x - y$

Vector valued function (e.g. velocity) $\vec{f} : D \subseteq \mathbb{R}^2 \to R \subseteq \mathbb{R}^2$

$(x,y) \mapsto \vec{f}(x,y)$

e.g. $D = R = \mathbb{R}^2$ $\quad \vec{f}(x,y) = \begin{pmatrix} 2x+y \\ x^2 \end{pmatrix}$

Back to 1d function: How to get a circle?



$x^2 + y^2 = R^2 \Rightarrow y = \pm\sqrt{R^2 - x^2} \Rightarrow$ 2 functions

implicit ~~function~~ curve

explicit function

Conclusion:

- Not possible to characterize circle with one explicit function (we need 2)
- Implicit curves are a nice way to define a curve

We want to use another way to define curves

Parametric curves

Parameter can be seen as time or angle

Idea for simple function $f(x) = x^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

$x = t$

$y = t^2 = x^2$ ✓

Or in general for functions $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ f(t) \end{pmatrix}$ $\quad x = t$, $y = f(t) = f(x)$ ✓

$\begin{pmatrix} x \\ y \end{pmatrix}$ ... coordinates at time t

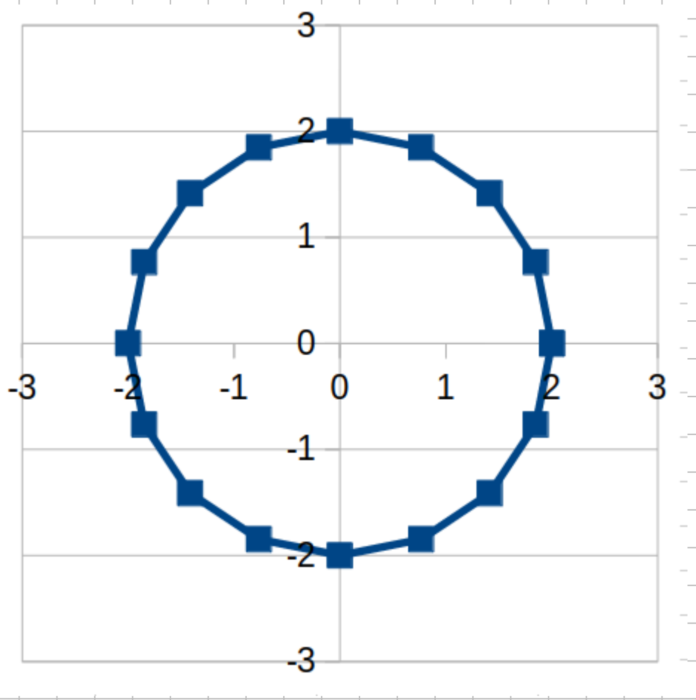
New idea: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ both coordinates are functions of time t

E.g. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2\cos t \\ 2\sin t \end{pmatrix}$ $\quad t \in [0, 2\pi)$

E.g. at time $t = \frac{\pi}{2}$ $\quad x\left(\frac{\pi}{2}\right) = 2 \cdot \cos\left(\frac{\pi}{2}\right) = 0$ $\quad y\left(\frac{\pi}{2}\right) = 2\sin\left(\frac{\pi}{2}\right) = 2$

Table with $t = 0, \frac{\pi}{8}, 2\frac{\pi}{8}, \dots, 16\frac{\pi}{8}$

t	x	y
0.00	2.00	0.00
0.39	1.85	0.77
0.79	1.41	1.41
1.18	0.77	1.85
1.57	0.00	2.00
1.96	-0.77	1.85
2.36	-1.41	1.41
2.75	-1.85	0.77
3.14	-2.00	0.00
3.53	-1.85	-0.77
3.93	-1.41	-1.41
4.32	-0.77	-1.85
4.71	0.00	-2.00
5.11	0.77	-1.85
5.50	1.41	-1.41
5.89	1.85	-0.77
6.28	2.00	0.00

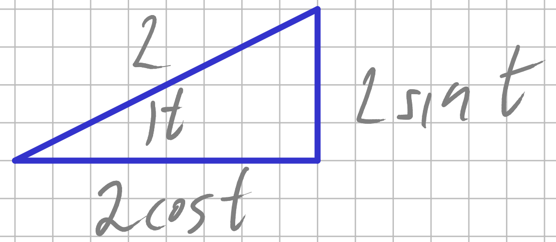


Circle with center $(0,0)$ radius 2

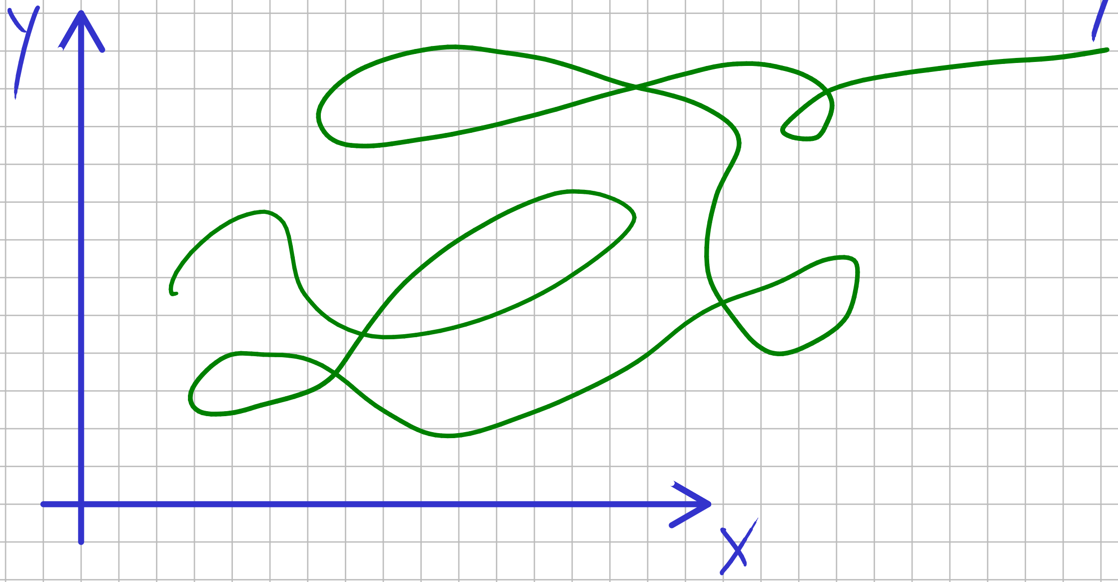
why?

$x = 2\cos t$

$y = 2\sin t$



Parametric curves ~> arbitrary curves in 2d / 3d



$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

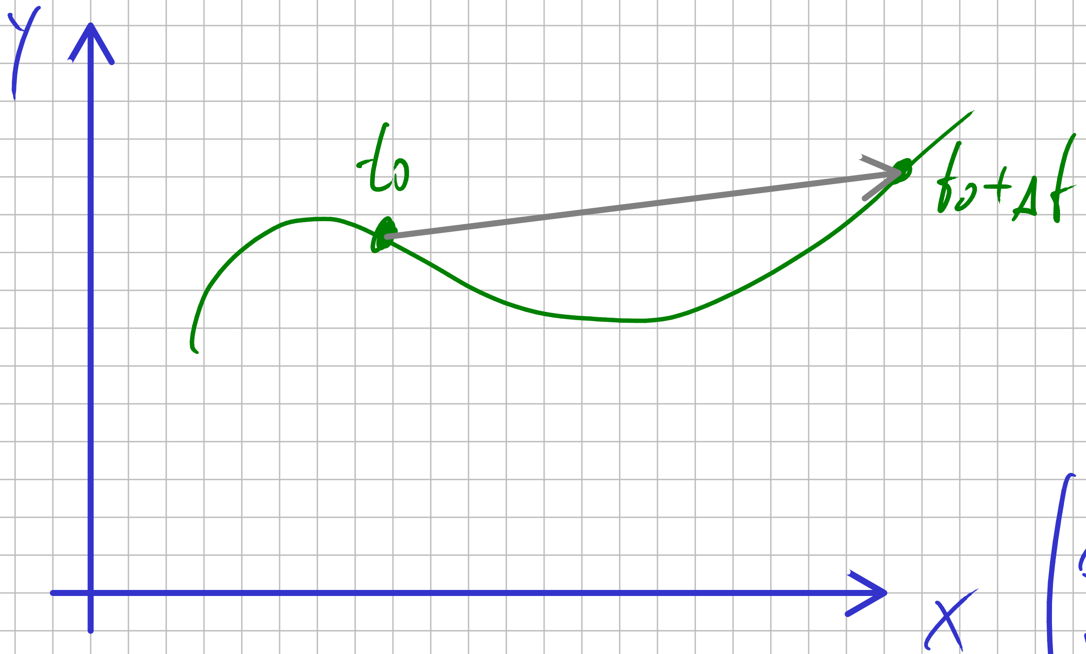
Example straight line $P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $Q = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\Rightarrow \vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\vec{r}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2t \\ 1+t \end{pmatrix} \quad t \in \mathbb{R}$$

For functions f, we know the tangent formula for x_0

$$y(x) = f(x_0) + f'(x_0)\cdot(x - x_0)$$

For parametric curves for t_0:



Distance between t_0 and $t_0+\Delta t$

$$\vec{r}(t_0+\Delta t) - \vec{r}(t_0)$$

Length per Δt:

$$\frac{\vec{r}(t_0+\Delta t) - \vec{r}(t_0)}{\Delta t} \rightsquigarrow \text{average velocity}$$

$$= \begin{pmatrix} \frac{x(t_0+\Delta t) - x(t_0)}{\Delta t} \\ \frac{y(t_0+\Delta t) - y(t_0)}{\Delta t} \end{pmatrix} \xrightarrow{\Delta t \to 0} \begin{pmatrix} x'(t_0) \\ y'(t_0) \end{pmatrix} = \begin{pmatrix} \dot{x}(t_0) \\ \dot{y}(t_0) \end{pmatrix}$$

$\Rightarrow$ Tangent $\quad \vec{g}(s) = \begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} + s \cdot \begin{pmatrix} \dot{x}(t_0) \\ \dot{y}(t_0) \end{pmatrix} \quad s \in \mathbb{R}$

Example a) $\vec{r}(t) = \begin{pmatrix} t \\ t^2 - t \end{pmatrix} \quad t_0 = 2 \qquad \dot{\vec{r}}(t) = \begin{pmatrix} 1 \\ 2t - 1 \end{pmatrix}$

$$\vec{g}(s) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad s \in \mathbb{R}$$

Get explicit function : $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \Big| \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$$-3x + y = -4 + 0 \Rightarrow y = 3x - 4$$

Example b) $\vec{r}(t) = \begin{pmatrix} t \sin(t) \\ \sin(t) \end{pmatrix}$ $\quad t_0 = \frac{\pi}{4}$

For a parametric curve a singular point is a point with zero velocity i.e. $\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Example a) $\vec{r}(t) = \begin{pmatrix} t^5 \\ t^2+1 \end{pmatrix}$ $\quad \dot{\vec{r}}(t) = \begin{pmatrix} 5t^4 \\ 2t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ for $t=0 \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) $\vec{r}(t) = \begin{pmatrix} \cos(t) \\ \sin(2t) \end{pmatrix}$ $\quad \dot{\vec{r}}(t) = \begin{pmatrix} -\sin(t) \\ 2 \cdot \cos(2t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad$ ↯

Curves in Polar coordinates

Cartesian coordinates x and y

polar coordinates r and φ
radius and angle

Sometimes it is simpler to use polar coordinates for functions $r = f(\varphi)$ instead of $y = f(x)$

We can get the parametric curve for $r = f(\varphi)$ by

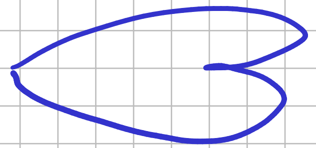
$$\vec{r}(\varphi) = \begin{pmatrix} x(\varphi) \\ y(\varphi) \end{pmatrix} = \begin{pmatrix} r(\varphi) \cdot \cos(\varphi) \\ r(\varphi) \cdot \sin(\varphi) \end{pmatrix}$$

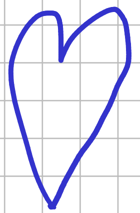
Example a) circle with center $(0,0)$ and radius 2

$r(\varphi) = 2$ constant $\varphi \in [0, 2\pi)$

b) $r(\varphi) = 2\varphi$ linear $\varphi \in [0, 4\pi)$ $\longrightarrow$ spiral

c) $r(\varphi) = 1 - \cos(\varphi)$ $\varphi \in [0, 2\pi)$ $\longrightarrow$



or  cardioid curve

Surfaces in Parameter form

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \rightarrow \text{curve in 2d}$$

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \quad \text{curve in 3d}$$

$$\vec{r}(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix}$$

s, t ... parameters

$s \in D_s \subseteq \mathbb{R}$

$t \in D_t \subseteq \mathbb{R}$

Special case $\quad x(s,t) = s$

$y(s,t) = t$

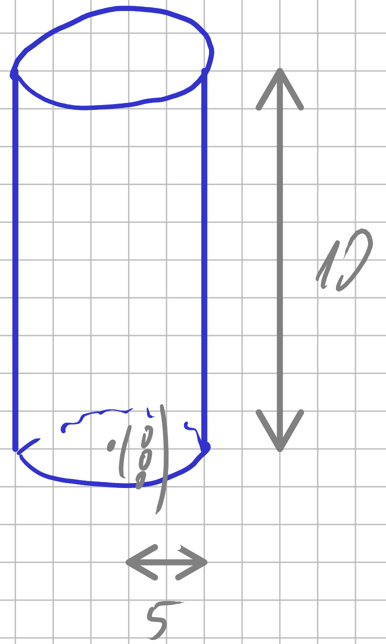
$z(s,t) = f(s,t) = f(x,y) \rightarrow$ function in 2d

Example

a) Plane with points $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\\1\end{pmatrix}$

$$\vec{r}(s,t) = \begin{pmatrix}1\\0\\0\end{pmatrix} + s\begin{pmatrix}0\\1\\0\end{pmatrix} + t\begin{pmatrix}0\\1\\1\end{pmatrix} \qquad s,t \in \mathbb{R}$$

b) Cylinder barrel radius 5 height 10

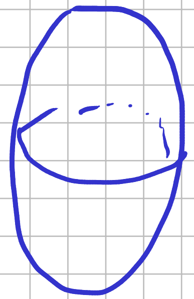


$$\vec{r}(\varphi, z) = \begin{pmatrix} 5\cos(\varphi) \\ 5\sin(\varphi) \\ z \end{pmatrix} \qquad \varphi \in [0, 2\pi) \qquad z \in [0, 10]$$

Cylinder Coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ z \end{pmatrix}$$

c) Sphere with radius 5



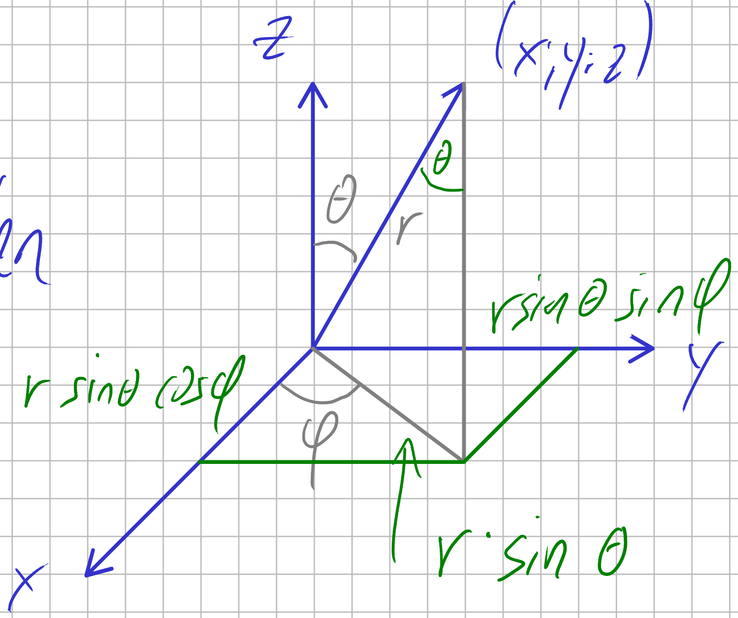
$$\vec{r}(\varphi, \theta) = \begin{pmatrix} 5\sin(\theta)\cos(\varphi) \\ 5\sin(\theta)\sin(\varphi) \\ 5\cos(\theta) \end{pmatrix}$$

$\varphi \in [0, 2\pi)$

$\theta \in [0, \pi]$

Spherical Coordinates

oben



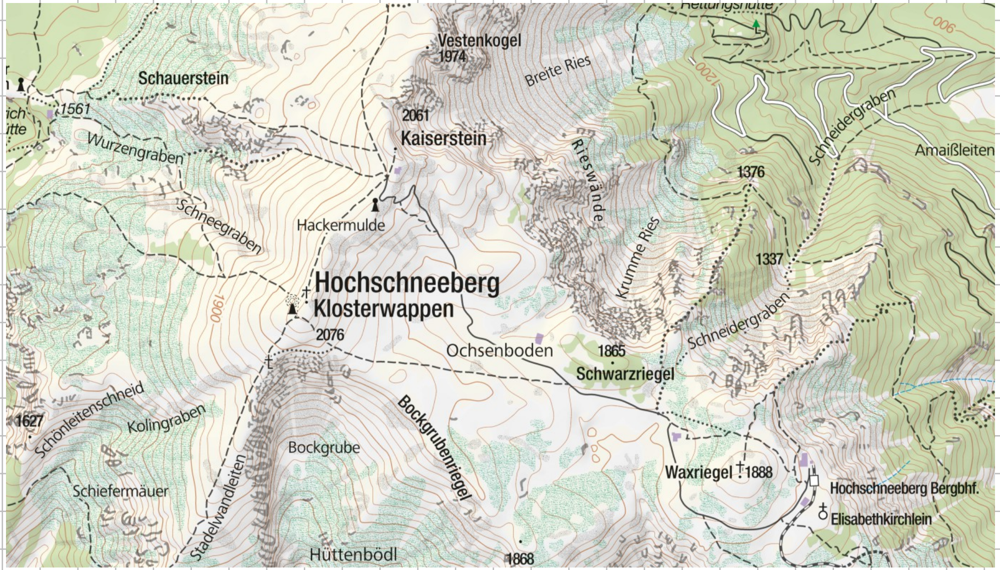
$$\vec{r}(\varphi, \theta) = \begin{pmatrix} r\sin(\theta)\cos(\varphi) \\ r\sin(\theta)\sin(\varphi) \\ r\sin(\theta) \end{pmatrix}$$

Contour plot and implicit curves

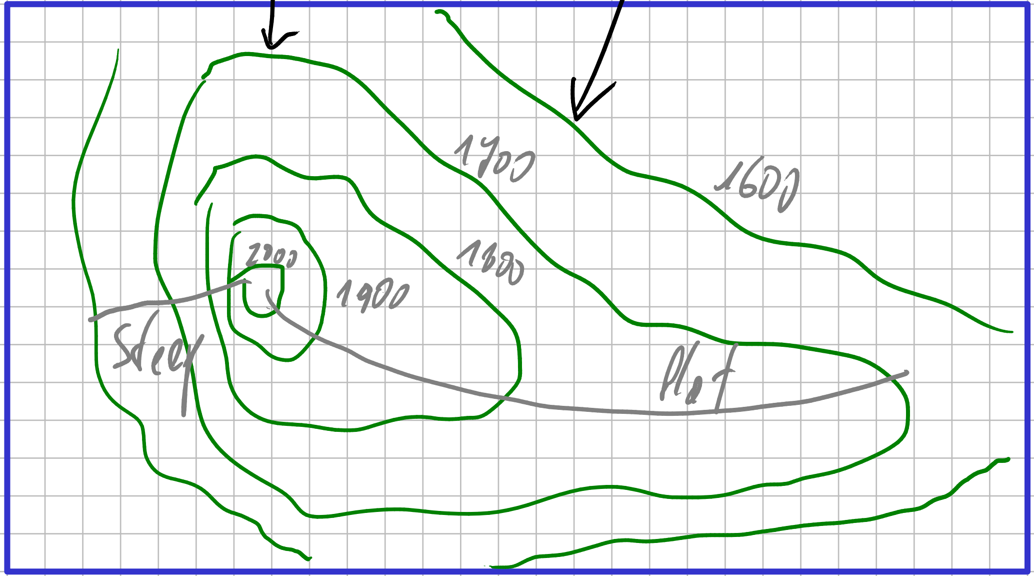
$$f : D \subseteq \mathbb{R}^2 \to R \subseteq \mathbb{R}$$
$$(x,y) \mapsto f(x,y)$$

function in 2d

Interpretation as see level



$f(x,y) = 1700$ $\quad$ $f(x,y) = 1600$



Implicit curve $f(x,y) = c$

Example

a) $x^2+y^2 = 4 \rightarrow$ circle with center $(0,0)$ radius 2

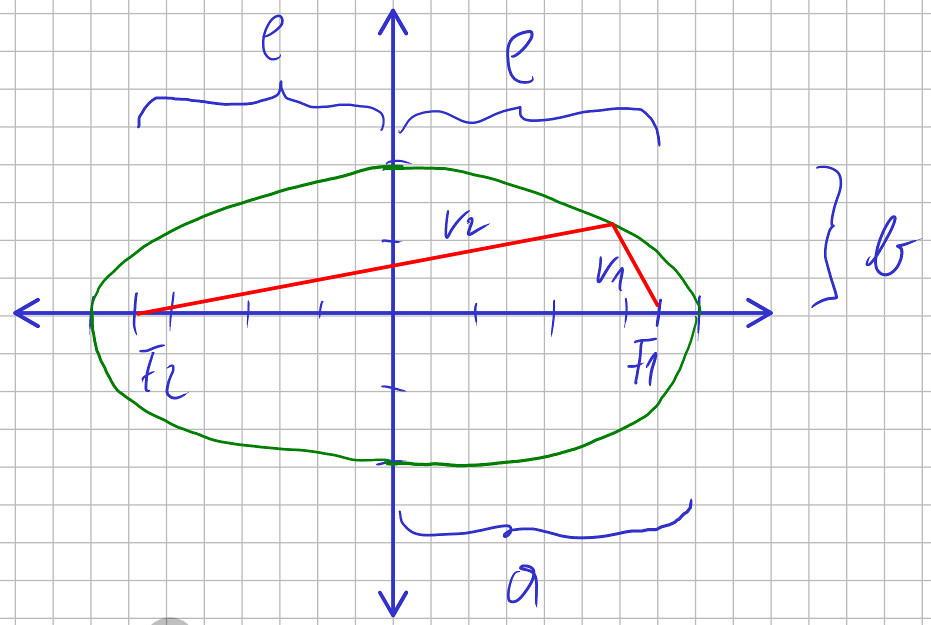
b) $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \rightarrow$ ellipse —〃— and $a=4$ $b=2$

Parameter form $\vec{r}(t) = \begin{pmatrix} 4\cos(t) \\ 2\sin(t) \end{pmatrix}$ $t \in [0, 2\pi)$

c) $\left(\frac{x}{4}\right)^2 - \left(\frac{y}{2}\right)^2 = 1 \rightarrow$ hyperbola —〃— —〃—

Parameter form $\vec{r}(t) = \begin{pmatrix} \pm 4 \, \frac{t^2+1}{2t} \\ 2 \, \frac{t^2-1}{2t} \end{pmatrix}$ $t \in \mathbb{R}$ or $\vec{r}(t) = \begin{pmatrix} \pm 4\cosh(t) \\ 2\sinh(t) \end{pmatrix}$ $t \in \mathbb{R}$

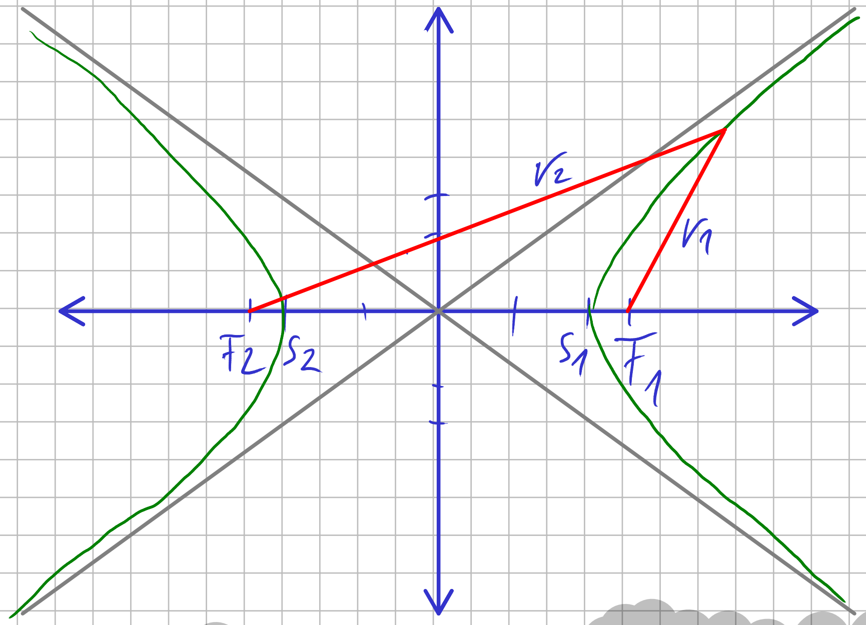
For interested readers, ellipse:

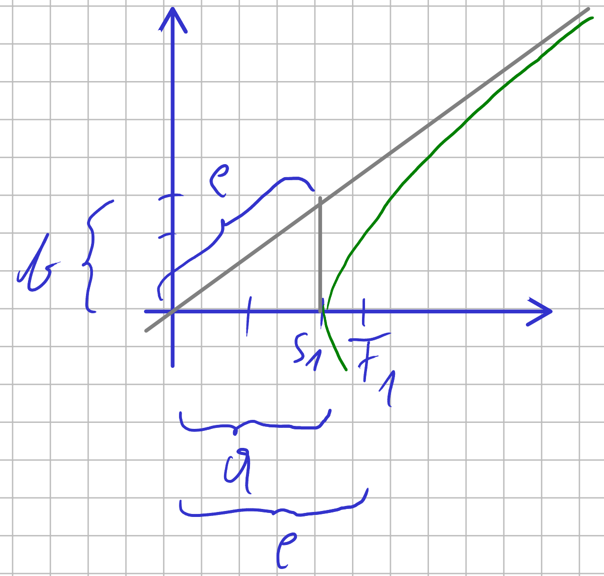


$$r_2 + r_1 = 2a$$

$$a^2 - b^2 = e^2$$

For interested readers, hyperbola:

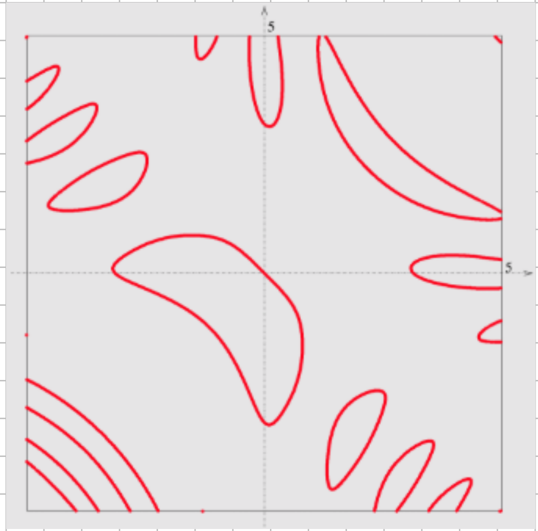


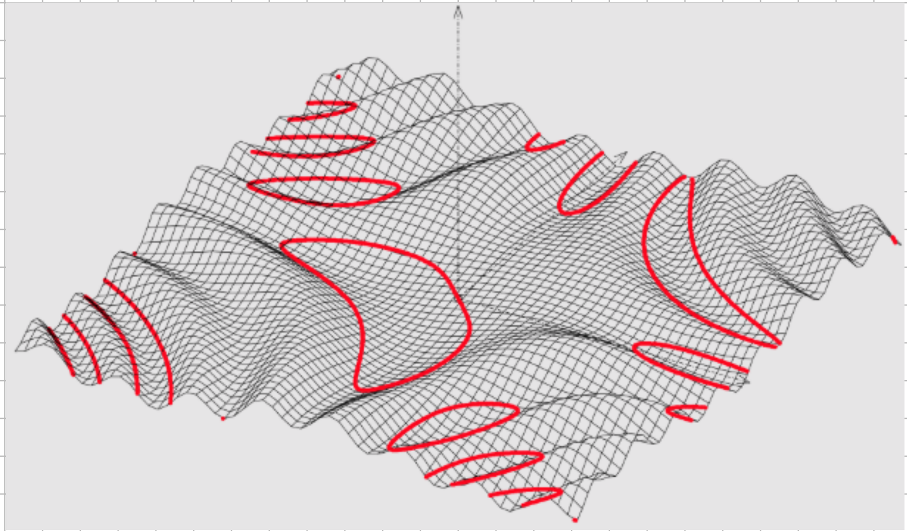


$$r_2 - r_1 = 2a$$

$$a^2 + b^2 = e^2$$

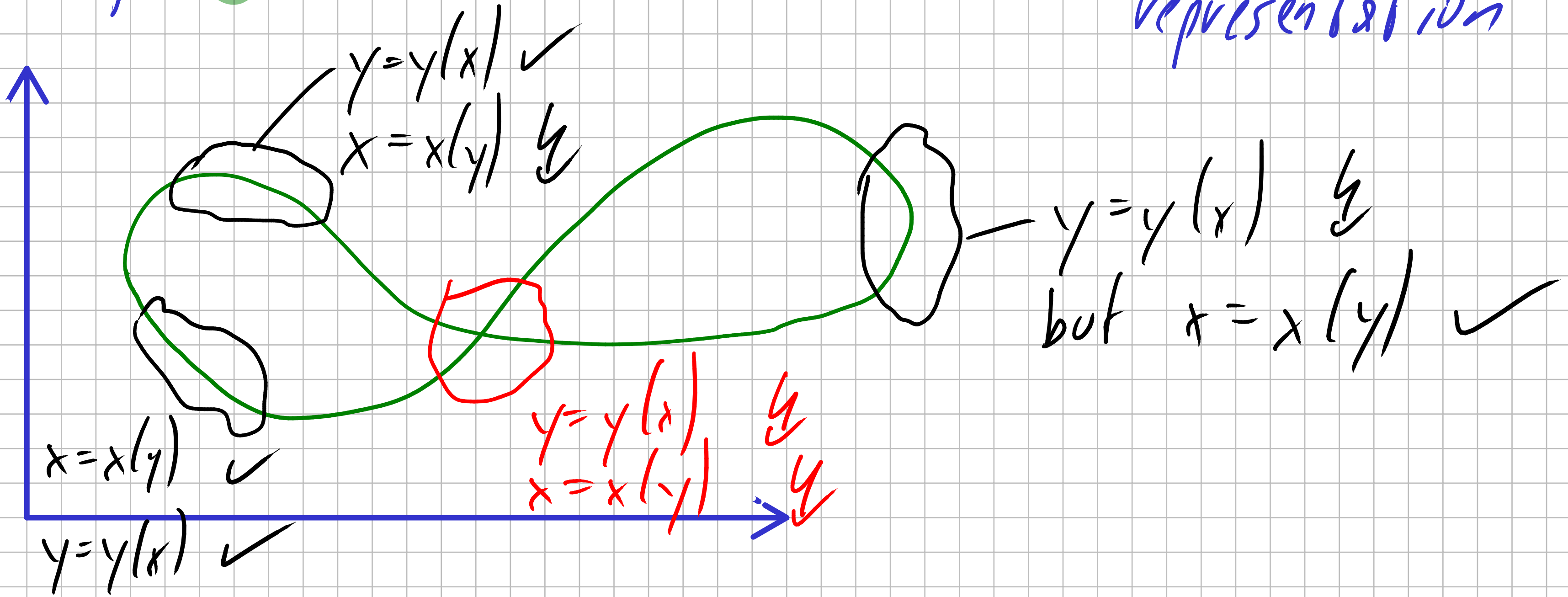
81) $\sin(x+y) - \cos(xy) + 1 = 0$





Tangent for implicit curves

Idea: locally use explicit (unknown) representation $y = y(x)$ or $x = x(y)$



$f(x,y) = c \quad \longrightarrow \quad f(x, y(x)) = c$ or $f(x(y), y) = c$ locally

How to get y' (or x') and how to get tangent?

Tangent for explicit functions: $\quad g(x) = y(x_0) + y'(x_0)(x - x_0)$

$$\frac{d}{dx} f(x, y(x)) = \frac{d}{dx} c \quad \rightsquigarrow \quad y'(x) = \ldots$$

Example $\quad f(x,y) = (x-5)^2 + (y-1)^2 = 1 \quad |$ circle center $= (5,1)$

radius $= \sqrt{1} = 1$

Find tangent for $x_0 = 4.5$

1) Find y_0: $\quad (4.5-5)^2 + (y-1)^2 = 1$

$$(y-1)^2 = 0.75$$

$$y - 1 = \pm\sqrt{0.75}$$

$$y = 1 \pm \sqrt{0.75} \qquad y_1 \approx 1.87 \qquad y_2 \approx 0.13$$

2) Get y'

$$\frac{d}{dx}(x-5)^2+(y-1)^2=\frac{d}{dx}1$$

$$2(x-5)+2(y-1)\cdot y'=0$$

$$y'=\frac{5-x}{y-1}$$

$x=4.5$ $\quad y\approx 1.87$ $\quad y'\approx 0.58$ (use exact value for y)

tangent: $g(x)\approx 1.87+0.58(x-4.5)$

$x=4.5$ $\quad y\approx 0.13$ $\quad y'\approx -0.58$ (use exact value for y)

tangent: $y(x)\approx 0.13-0.58(x-4.5)$

Faster way: $y' = \frac{-f_x}{f_y} = \frac{5-x}{y-1}$

$$f(x,y) = (x-5)^2 + (y-1)^2$$

$$f_x(x,y) = 2(x-5)$$

$$f_y(x,y) = 2(y-1)$$

Our approach works if f_y (denominator) is not zero

If f_y is zero but f_x is not $\Rightarrow$ same idea with $x = x(y) \Rightarrow x' = \frac{-f_y}{f_x}$

Only problem if $f_x = f_y = 0$ $\Rightarrow$ we call this the singular points for an implicit curve

Singular points for parametric curves are different! (no velocity)

Example

a) $x^3 - y^2 = 0$

Cusp

$f = x^3 - y^2$

$f_x = 3x^2 \overset{!}{=} 0 \Rightarrow x = 0$

$f_y = -2y \overset{!}{=} 0 \Rightarrow y = 0$

$\Rightarrow (0,0)$

b) $x^2(x+1) - y^2 = 0 \longrightarrow$ node

c) $x^4 - y^2 = 0 \longrightarrow$ tacnode

Tangent Plane for Parametric Surfaces

For a 2d function $f(x,y)$ we know the tangent plane

$$g(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

or as vector

$$\vec{g}(s,t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ f(x_0,y_0) \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ f_x(x_0,y_0) \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ f_y(x_0,y_0) \end{pmatrix} \qquad s,t \in \mathbb{R}$$

Example $f(x,y) = 2xy \qquad x_0 = 2 \quad y_0 = 1$

$f(x_0, y_0) = 4$

$f_x(x,y) = 2y \qquad f_x(2,1) = 2$

$f_y(x,y) = 2x \qquad f_y(2,1) = 4$

$$g(x,y) = 4 + 2(x-2) + 4(y-1) = -4 + 2x + 4y$$

$$\vec{g}(s,t) = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \qquad s,t \in \mathbb{R}$$

Is this the same plane? normal vector $\vec{n} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \quad / \cdot \begin{pmatrix} -2 \\ -4 \\ 1 \end{pmatrix}$$

$$-2x - 4y + z = -4 - 4 + 4 = -4$$

$$z = -4 + 2x + 4y \quad \checkmark$$

And for parametric surfaces $\vec{r}(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ y(s,t) \end{pmatrix}$ $\quad s \in D_s \quad t \in D_t$

$$\vec{q}(s,t) = \vec{r}(s_0,t_0) + s \begin{pmatrix} x_s(s_0,t_0) \\ y_s(s_0,t_0) \\ z_s(s_0,t_0) \end{pmatrix} + t \begin{pmatrix} x_t(s_0,t_0) \\ y_s(s_0,t_0) \\ z_s(s_0,t_0) \end{pmatrix} \quad s,t \in \mathbb{R}$$

For the special case $x = s$ $y = t$ $z = f(x,y)$ we get the original formula

Example $\vec{r}(\varphi, z) = \begin{pmatrix} 5\cos(\varphi) \\ 5\sin(\varphi) \\ z \end{pmatrix}$ $\varphi \in [0, 2\pi)$ $z \in [0, 10]$

$\varphi_0 = \frac{\pi}{2}$ $z_0 = 5$

$\vec{r}\left(\frac{\pi}{2}, 5\right) = \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix}$ $\vec{r}_\varphi(\varphi, z) = \begin{pmatrix} -5\sin(\varphi) \\ 5\cos(\varphi) \\ 0 \end{pmatrix}$ $\vec{r}_\varphi\left(\frac{\pi}{2}, 5\right) = \begin{pmatrix} -5 \\ 0 \\ 0 \end{pmatrix}$

$\vec{r}_z(\varphi, z) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\vec{r}_z\left(\frac{\pi}{2}, 5\right) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\Rightarrow \vec{g}(s,t) = \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} + s\begin{pmatrix} -5 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $s, t \in \mathbb{R}$

$$\vec{n} = \begin{pmatrix} -5 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} + s \begin{pmatrix} -5 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Big| \cdot \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$$

$$\Rightarrow 5y = 25 \quad \Rightarrow \quad y = 5$$

Implicit Surfaces

$f(x,y) = c \rightarrow$ implicit curve

$f(x,y,z) = c \rightarrow$ implicit surface

Tangental plane for x_0, y_0, z_0

$$f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0$$

If $f(x,y,z) = \tilde{f}(x,y) - z \Rightarrow f_x = \tilde{f}_x \quad f_y = \tilde{f}_y \quad f_z = -1$

$\Rightarrow$ standard formula

$$\underbrace{f_x(x_0,y_0,z_0)}_{\tilde{f}_x(x_0,y_0)}(x-x_0) + \underbrace{f_y(x_0,y_0,z_0)}_{\tilde{f}_y(x_0,y_0)}(y-y_0) + \underbrace{f_z(x_0,y_0,z_0)}_{-1}(z-z_0) = 0$$

$$z = z_0 + \tilde{f}_x(x_0,y_0)(x-x_0) + \tilde{f}_y(x_0,y_0)(y-y_0)$$

Example $\quad x^2+y^2+z^2=1 \quad$ sphere with center $(0,0,0)$ radius 1

$x_0 = \frac{1}{2} \qquad y_0 = \frac{1}{2} \qquad z_0 = ?$

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + z^2 = 1 \quad \Rightarrow \quad z = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$$

$$\left.\begin{array}{ll} f_x(x,y,z) = 2x & f_x\left(\frac{1}{2},\frac{1}{2},\frac{\sqrt{2}}{2}\right) = 1 \\ f_y(x,y,z) = 2y & f_y\left(\frac{1}{2},\frac{1}{2},\frac{\sqrt{2}}{2}\right) = 1 \\ f_z(x,y,z) = 2z & f_z\left(\frac{1}{2},\frac{1}{2},\pm\frac{\sqrt{2}}{2}\right) = \pm\sqrt{2} \end{array}\right\}$$

$$1\cdot\left(x-\frac{1}{2}\right) + 1\cdot\left(y-\frac{1}{2}\right) \pm\sqrt{2}\left(z\mp\frac{\sqrt{2}}{2}\right) = 0$$

$$x+y-2 = \mp\sqrt{2}\,z$$

$$t_1: \quad z = \frac{x+y-2}{-\sqrt{2}} = \frac{\sqrt{2}}{2}(2-x-y)$$

$$t_2: \quad z = \frac{x+y-2}{\sqrt{2}} = \frac{\sqrt{2}}{2}(-2+x+y)$$