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Calculus
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Session 6

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Local Maximum/Minimum Values

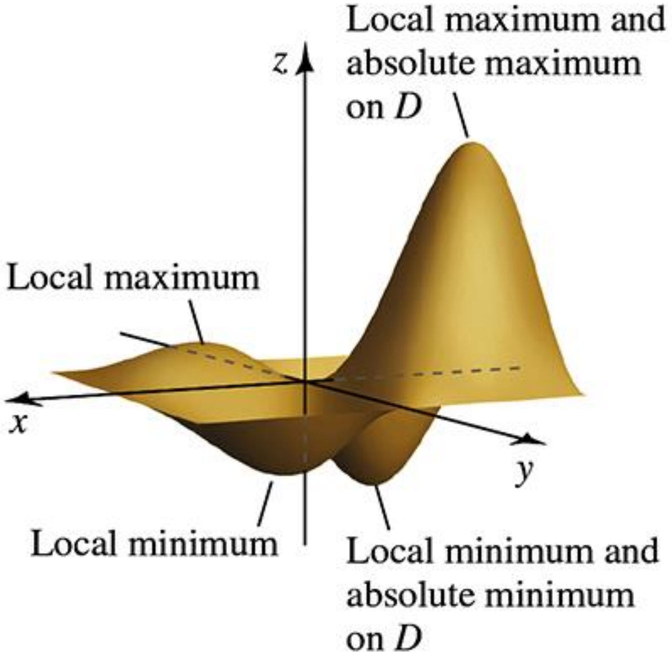
Definition

Suppose (a, b) is a point in a region R on which f is defined.

If $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a local maximum value of f.

If $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a local minimum value of f.

Local maximum and local minimum values are also called local extreme values or local extrema.



Derivatives and Local Maximum/Minimum Values

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Theorem

If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.

Critical Point

Definition

An interior point (a, b) in the domain of f is a critical point of f if either

- $f_x(a, b) = f_y(a, b) = 0$, *or*
- *At least* one of the partial derivatives f_x and f_y does not exist at (a, b).

Maximum/Minimum Problems

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Example

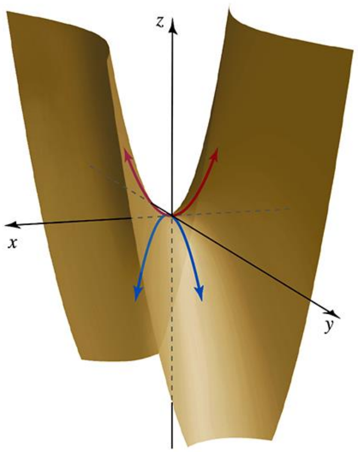
Find the critical points of $f(x, y) = xy(x-2)(y+3)$.

Saddle Point

Definition

Consider a function f that is differentiable at a critical point (a, b).

Then f has a saddle point at (a, b) if, in every open disk centered at (a, b), there are points (x, y) for which $f(x, y) > f(a, b)$ and points for which $f(x, y) < f(a, b)$.



The hyperbolic paraboloid $z = x^2 - y^2$ has a saddle point at (0, 0).

Second Derivative Test

Theorem

Suppose the second partial derivatives of f are continuous throughout an open disk centered at the point (a, b), where $f_x(a, b) = f_y(a, b) = 0$.

Let $D(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - (f_{xy}(x, y))^2$.

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a **local maximum value** at (a, b).
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a **local minimum value** at (a, b).
3. If $D(a, b) < 0$, then f has a **saddle point** at (a, b).
4. If $D(a, b) = 0$, then the test is **inconclusive.**

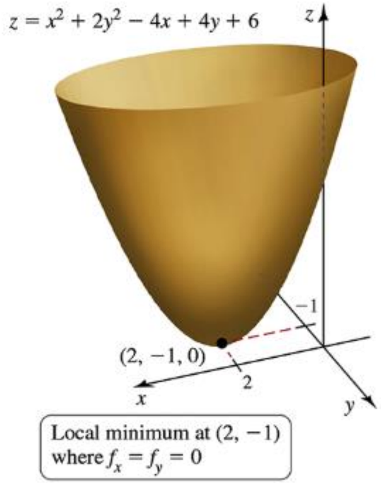
Second Derivative Test

- $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - (f_{xy}(x,y))^2 \rightarrow$ the discriminant of f
- The 2×2 determinant of the Hessian matrix $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$, where $f_{xy} = f_{yx}$.
- $D(x,y) > 0 \rightarrow$ the surface has the same general behavior in all directions near (a,b); either the surface rises in all directions or it falls in all directions.
- $D(x,y) = 0 \rightarrow$ the test is inconclusive: (a,b) could correspond to a local maximum, a local minimum, or a saddle point.

Analyzing critical points

Example

Use the Second Derivative Test to classify the critical points of $f(x, y) = x^2 + 2y^2 - 4x + 4y + 6$.



Analyzing critical points

$(x-1)' = 1$

Example

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$

Use the Second Derivative Test to classify the critical points of $f(x, y) = xy(x - 2)(y + 3)$.

Critical points: $(0,0)$, $(2,0)$, $(1,-\frac{3}{2})$, $(0,-3)$, $(2,-3)$

$f_x = 2y\,(x-1)(y+3)$

$f_y = x \cdot (x-2) \cdot (2y+3)$

$f_{xx} = 2y(y+3)$

$f_{yy} = 2x(x-2)$

$(2yx - 2y) \cdot (y+3) =$

$2y^2x + 6yx - 2y^2 - 6y = f_x$

$f_{xy} = 4yx + 6x - 4y - 6$

$= 6(x-1) + 4y(x-1)$

$= (x-1) \cdot (6+4y) = 2 \cdot (x-1) \cdot (2y+3)$

Absolute Maximum and Minimum Values

Definition

Let f be defined on a set R in $\mathbb{R}^2$ containing the point (a, b).

If $f(a, b) \geq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute maximum value** of f on R.

If $f(a, b) \leq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute minimum value** of f on R.

Absolute Maximum and Minimum Values

Procedure: Finding Absolute Maximum/Minimum Values on Closed Bounded Sets

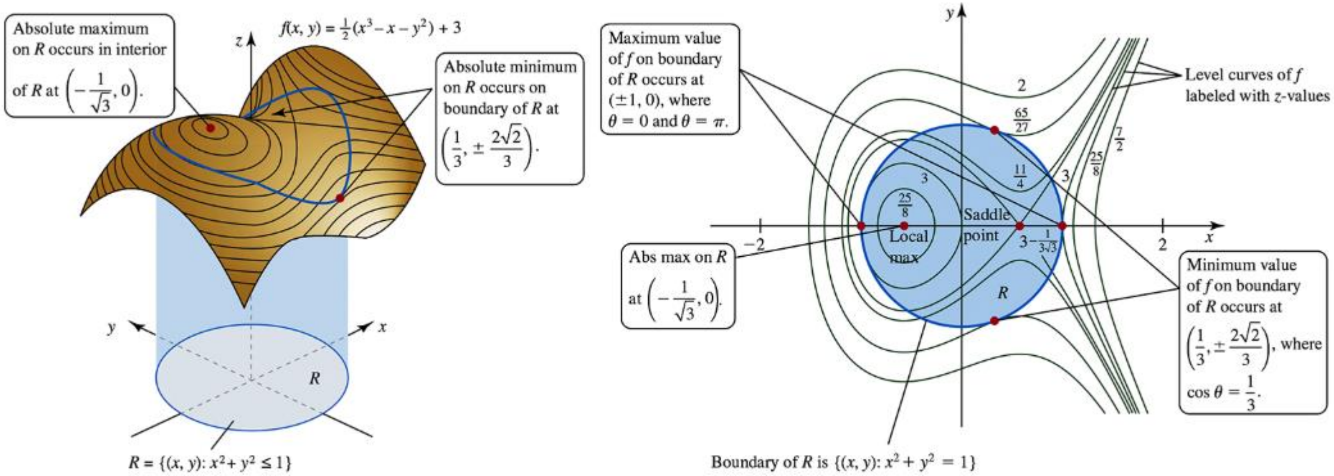
Let f be continuous on a closed bounded set R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R:

1. Determine the values of f at all critical points in R.
2. Find the maximum and minimum values of f on the boundary of R .
3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on R, and the least function value found in Steps 1 and 2 is the absolute minimum value of f on R.

Absolute Maximum and Minimum Values

Example

Find the absolute maximum and minimum values of $f(x,y) = 12(x^3 - x - y^2) + 3$ on the region $R = \{(x,y)\colon x^2 + y^2 \leq 1\}$ (the closed disk centered at $(0,0)$ with radius 1).



Thank you!