



**FACHHOCHSCHULE
WIENER NEUSTADT**
Austrian Network for Higher Education

Calculus

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Session 6

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Local Maximum/Minimum Values

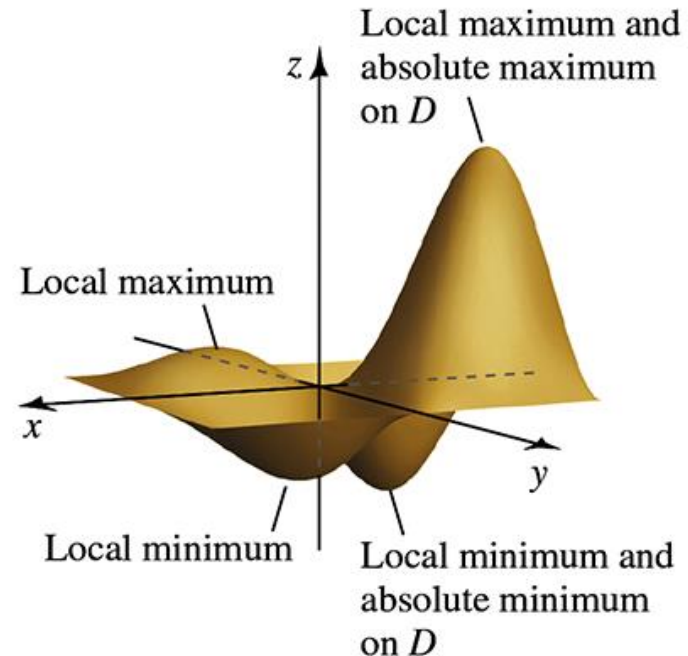
Definition

Suppose (a, b) is a point in a region R on which f is defined.

If $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b) , then $f(a, b)$ is a local maximum value of f .

If $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b) , then $f(a, b)$ is a local minimum value of f .

Local maximum and local minimum values are also called local extreme values or local extrema.





Derivatives and Local Maximum/Minimum Values

Theorem

If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b) , then $f_x(a, b) = f_y(a, b) = 0$.

Critical Point

Definition

An interior point (a, b) in the domain of f is a critical point of f if either

- $f_x(a, b) = f_y(a, b) = 0$, or
- At least one of the partial derivatives f_x and f_y does not exist at (a, b) .



Maximum/Minimum Problems

Example

Find the critical points of $f(x, y) = xy(x - 2)(y + 3)$.

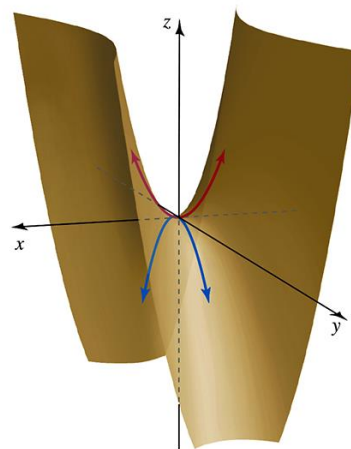
Saddle Point



Definition

Consider a function f that is differentiable at a critical point (a, b) .

Then f has a saddle point at (a, b) if, in every open disk centered at (a, b) , there are points (x, y) for which $f(x, y) > f(a, b)$ and points for which $f(x, y) < f(a, b)$.



The hyperbolic paraboloid $z = x^2 - y^2$ has a saddle point at $(0, 0)$.

Second Derivative Test

Theorem

Suppose the second partial derivatives of f are continuous throughout an open disk centered at the point (a, b) , where $f_x(a, b) = f_y(a, b) = 0$.

Let $D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - (f_{xy}(x, y))^2$.

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a **local maximum value** at (a, b) .
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a **local minimum value** at (a, b) .
3. If $D(a, b) < 0$, then f has a **saddle point** at (a, b) .
4. If $D(a, b) = 0$, then the test is **inconclusive**.

Second Derivative Test

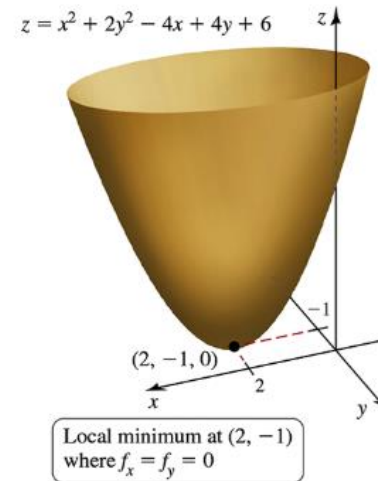


- $D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - (f_{xy}(x, y))^2 \rightarrow$ the discriminant of f
- The 2×2 determinant of the Hessian matrix $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$, where $f_{xy} = f_{yx}$.
- $D(x, y) > 0 \rightarrow$ the surface has the same general behavior in all directions near (a, b) ; either the surface rises in all directions or it falls in all directions.
- $D(x, y) = 0 \rightarrow$ the test is inconclusive: (a, b) could correspond to a local maximum, a local minimum, or a saddle point.

Analyzing critical points

Example

Use the Second Derivative Test to classify the critical points of $f(x, y) = x^2 + 2y^2 - 4x + 4y + 6$.





Analyzing critical points

Example

Use the Second Derivative Test to classify the critical points of $f(x, y) = xy(x - 2)(y + 3)$.

Absolute Maximum and Minimum Values

Definition

Let f be defined on a set R in $\mathbb{R}^2$ containing the point (a, b) .

If $f(a, b) \geq f(x, y)$ for every (x, y) in R , then $f(a, b)$ is an **absolute maximum value** of f on R .

If $f(a, b) \leq f(x, y)$ for every (x, y) in R , then $f(a, b)$ is an **absolute minimum value** of f on R .

Absolute Maximum and Minimum Values

Procedure: Finding Absolute Maximum/Minimum Values on Closed Bounded Sets

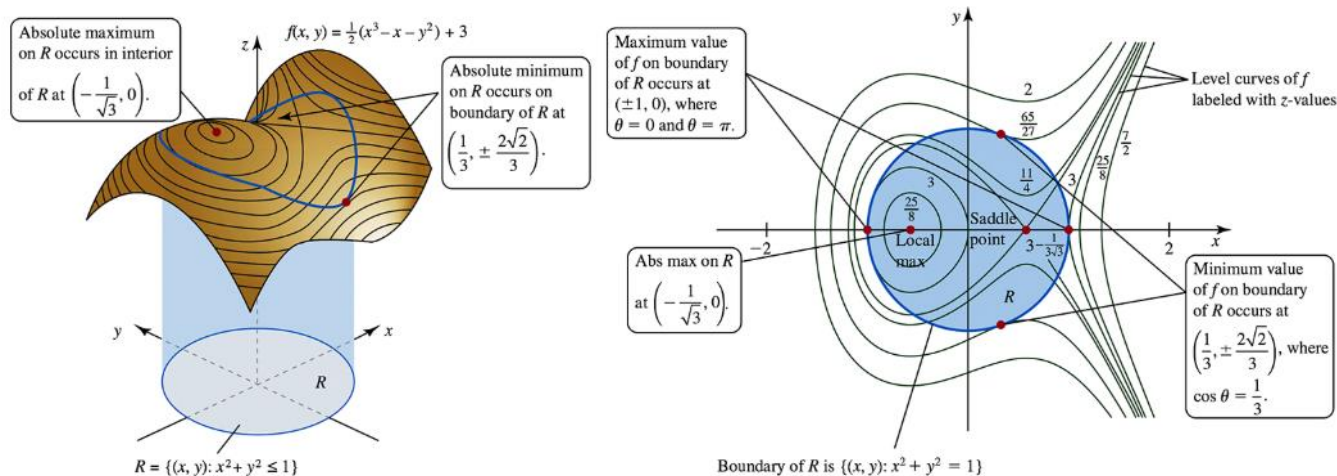
Let f be continuous on a closed bounded set R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R :

1. Determine the values of f at all critical points in R .
2. Find the maximum and minimum values of f on the boundary of R .
3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on R , and the least function value found in Steps 1 and 2 is the absolute minimum value of f on R .

Absolute Maximum and Minimum Values

Example

Find the absolute maximum and minimum values of $f(x, y) = 12(x^3 - x - y^2) + 3$ on the region $R = \{(x, y): x^2 + y^2 \leq 1\}$ (the closed disk centered at $(0, 0)$ with radius 1).





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Thank you!
