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Calculus
CSCI 2025
Session 5

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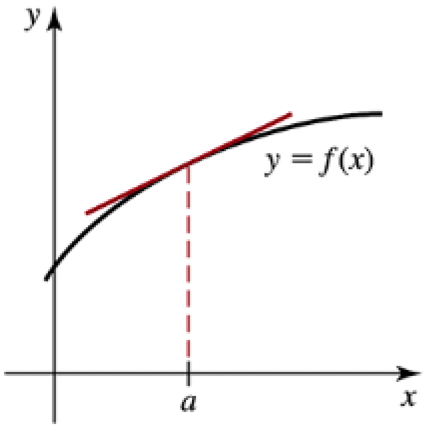
Overview

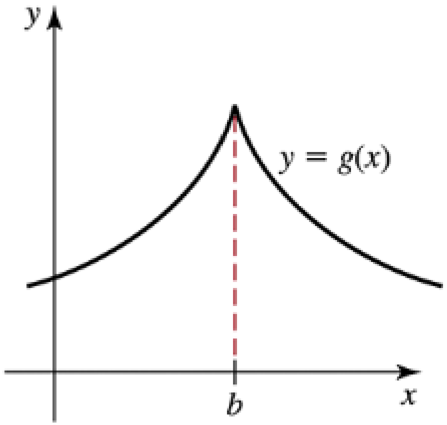
- Tangent Planes
- Linear Approximation

Tangent Planes

What have we seen until now?



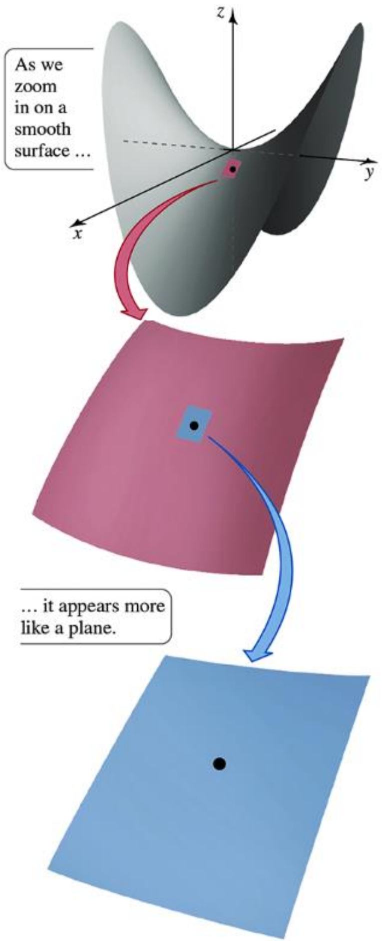
f differentiable at $a \Longrightarrow$ tangent line at $(a, f(a))$



g not differentiable at $b \Longrightarrow$ no tangent line at $(b, g(b))$

Tangent Planes

What will we see now?



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Tangent Planes

Tangent Planes for $F(x, y, z) = 0$

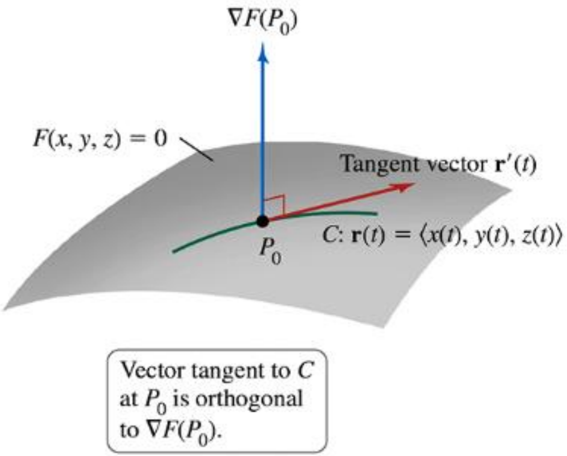
- To find an equation of the tangent plane, consider a smooth curve $C: r(t) = \langle x(t), y(t), z(t) \rangle$ that lies on the surface $F(x, y, z) = 0$.
- Because the points of C lie on the surface, we have $F(x(t), y(t), z(t)) = 0$.
- Differentiating both sides of this equation with respect to t reveals a useful relationship.
- The derivative of the right side is 0.

$$\begin{aligned}\frac{d}{dt}\Big(F\big(x(t), y(t), z(t)\big)\Big) &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} \\ &= \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \\ &= \nabla F(x, y, z) \cdot r'(t)\end{aligned}$$

Tangent Planes

Tangent Planes for $F(x, y, z) = 0$

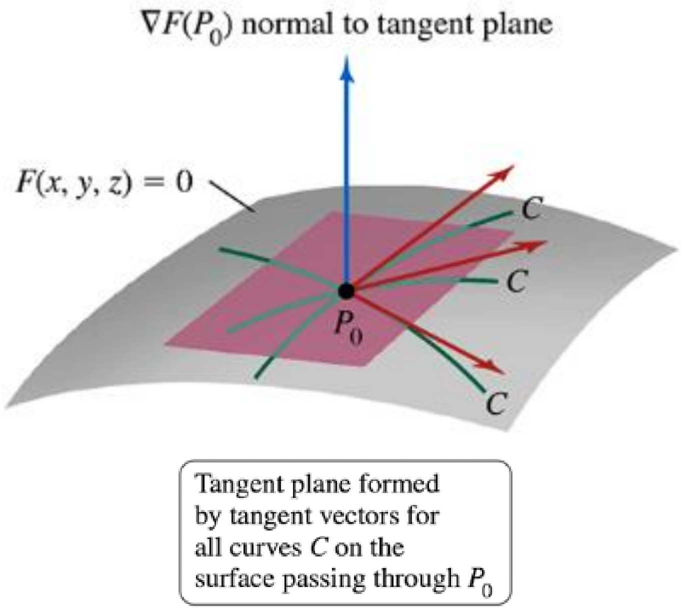
- Therefore $\nabla F(x, y, z) \cdot r'(t) = 0$ and at any point on the curve.
- The tangent vector $r'(t)$ is orthogonal to the gradient.



Tangent Planes

Tangent Planes for $F(x, y, z) = 0$

- $P_0(a, b, c)$ on the surface
- $\nabla F(x, y, z) \neq 0$
- C any smooth curve on the surface passing through P_0
- $\nabla F(x, y, z) \cdot \langle x - a, y - b, z - c \rangle = 0$



Tangent Planes

Definition

Equation of the Tangent Plane for $F(x, y, z) = 0$

Let F be differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq 0$. The plane tangent to the surface $F(x, y, z) = 0$ at P_0, called the **tangent plane,** is the plane passing through P_0 orthogonal to $\nabla F(a, b, c)$.

An equation of the tangent plane is

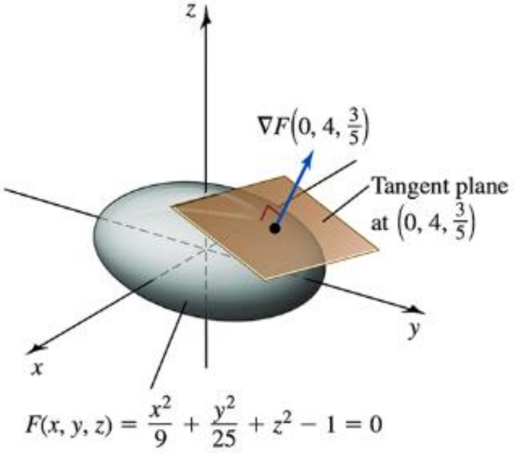
$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0.$$

Tangent Planes

Example

Consider the ellipsoid $F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0$.

- Find an equation of the plane tangent to the ellipsoid at $(0,4,\frac{3}{5})$.
- At what points on the ellipsoid is the tangent plane horizontal?



Tangent Planes

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Tangent Plane for $z = f(x, y)$

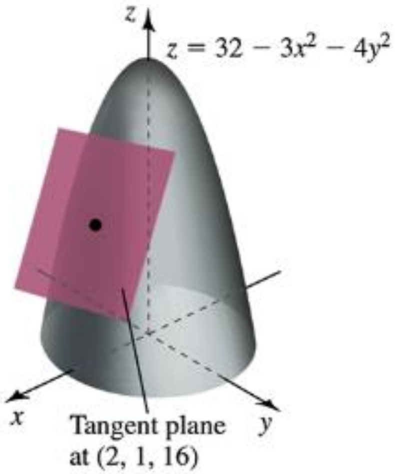
Let f be differentiable at the point (a, b). An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

Tangent Planes

Tangent Plane for $z = f(x, y)$

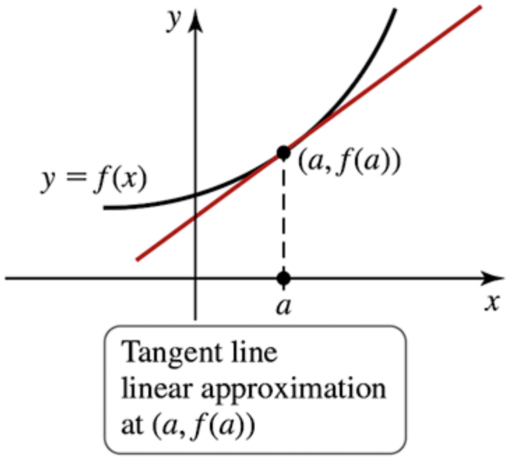
Find an equation of the plane tangent to the paraboloid $z = f(x, y) = 32 - 3x^2 - 4y^2$ at $(2, 1, 16)$.



Tangent Planes

Linear Approximation

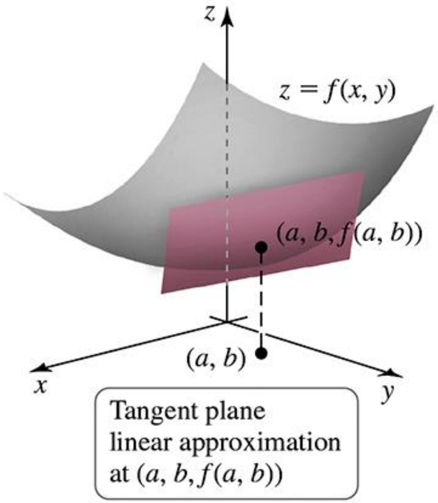
- One-variable case: if f is differentiable at a, the equation of the line tangent to the curve $y = f(x)$ at the point $(a, f(a))$ is $L(x) = f(a) + f'(a)(x - a)$.
- The tangent line gives an approximation to the function. At points near a, we have $f(x) \approx L(x)$.



Tangent Planes

Linear Approximation

- Two-variable case: if f is differentiable at (a, b), an equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is $L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$.
- The tangent plane is the linear approximation to f at (a, b). At points near (a, b), we have $f(x, y) \approx L(x, y)$.

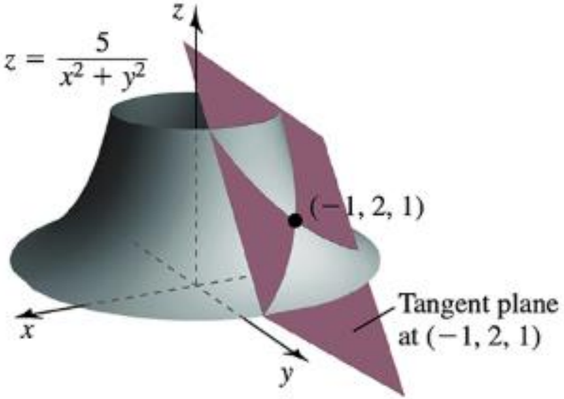


Tangent Planes

Example

Let $f(x,y) = \frac{5}{x^2+y^2}$.

- Find the linear approximation to the function at the point $(-1,2,1)$.
- Use the linear approximation to estimate the value of $f(-1.05, 2.1)$.



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Thank you!