Calculus
CSCI 2025

Session 4

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Overview

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- Partial Derivatives
- Directional Derivatives
- Gradient

Chain Rule

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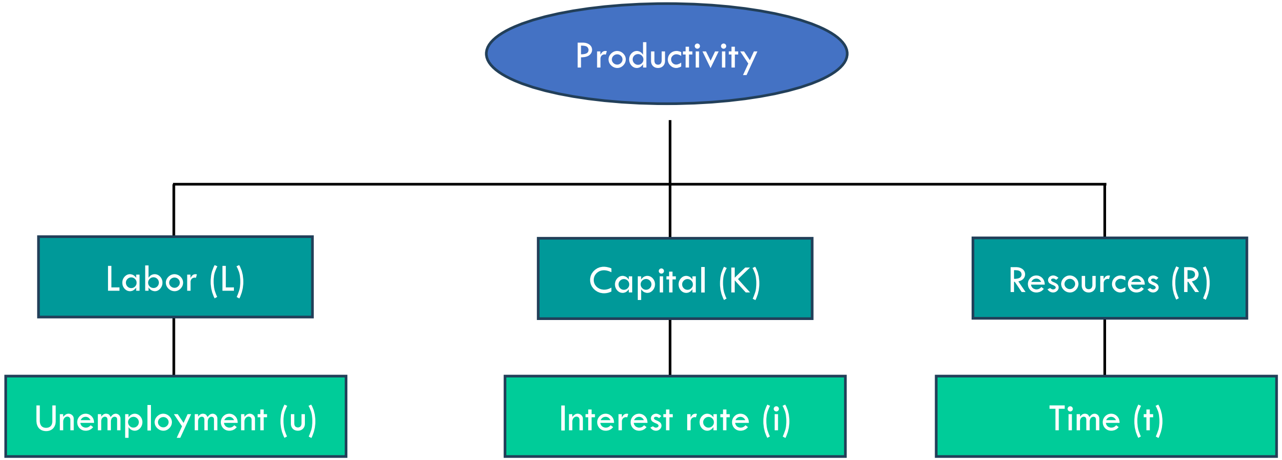


Chain Rule (One Independent Variable)

Theorem

z is a function of both x and y

Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

x and y depend only on t

$z = f\big(x(t), y(t)\big)$

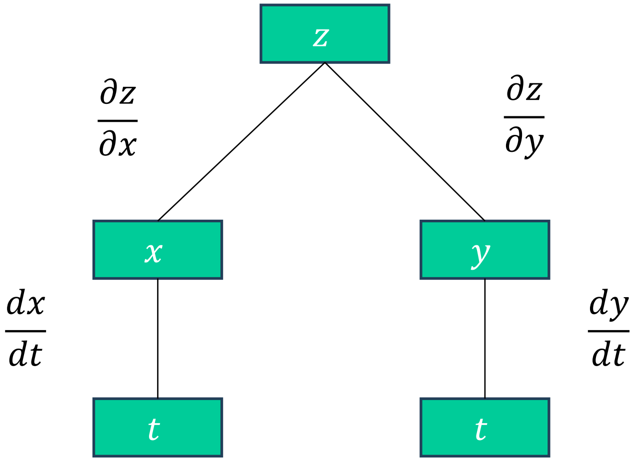
z: dependent variable

t: sole independent variable

x and y: intermediate variables

Chain Rule (One Independent Variable)

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$



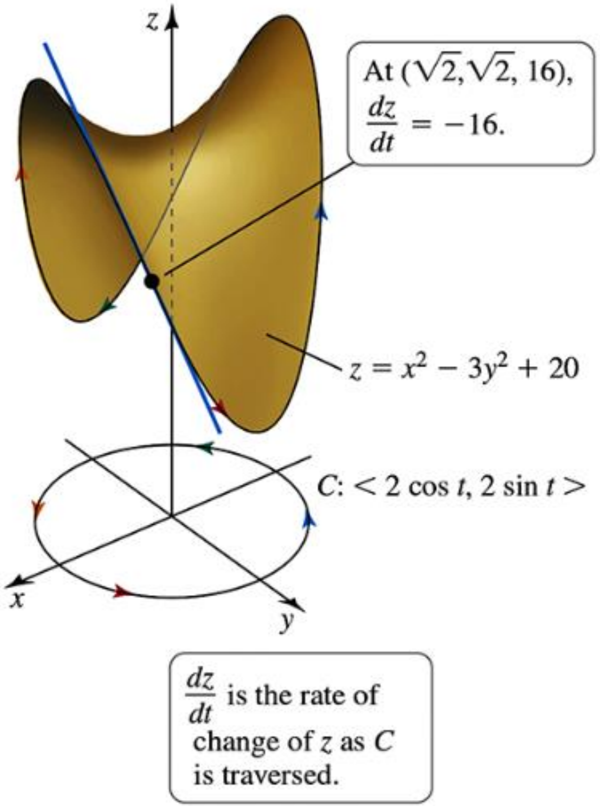
Chain Rule (One Independent Variable)

Example

Let $z = x^2 - 3y^2 + 20$, where $x = 2cost$ and $y = 2sint$.

a) Find $\frac{dz}{dt}$ and evaluate it at $t = \frac{\pi}{4}$.

b) Interpret the result geometrically.



Chain Rule (Two Independent Variables)

Theorem

Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Chain Rule (Two Independent Variables)

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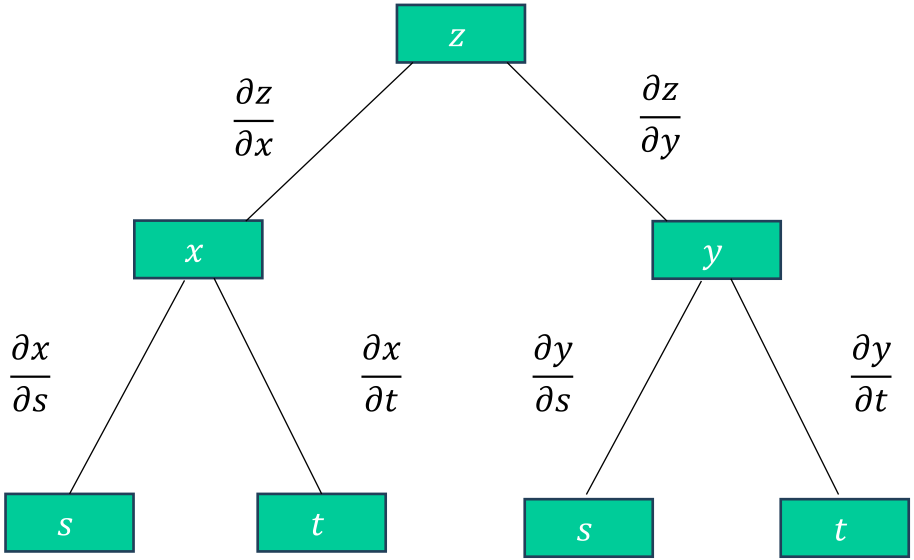
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



Chain Rule (Two Independent Variables)

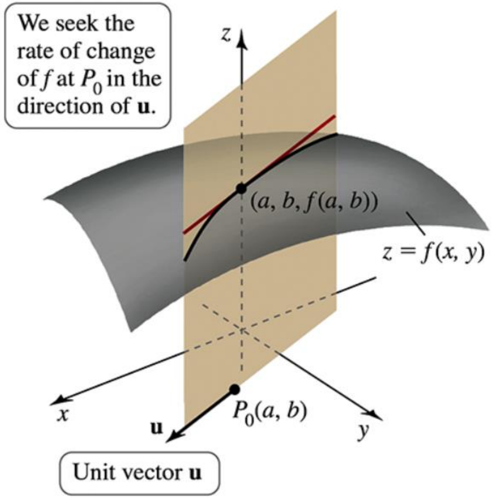
Example

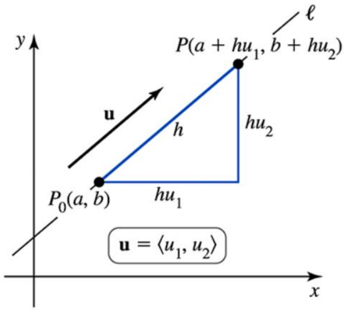
Let $z = sin2xcos3y$, where $x = s + t$ and $y = s - t$.

Evaluate $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Directional Derivatives

- Let $(a, b, f(a, b))$ be a point on the surface $z = f(x, y)$ and let u be a unit vector in the xy-plane.
- Our aim: find the rate of change of f in the direction u at $P_0(a, b)$.
- This rate of change is a combination of $f_x(a, b)$ and $f_y(a, b)$.





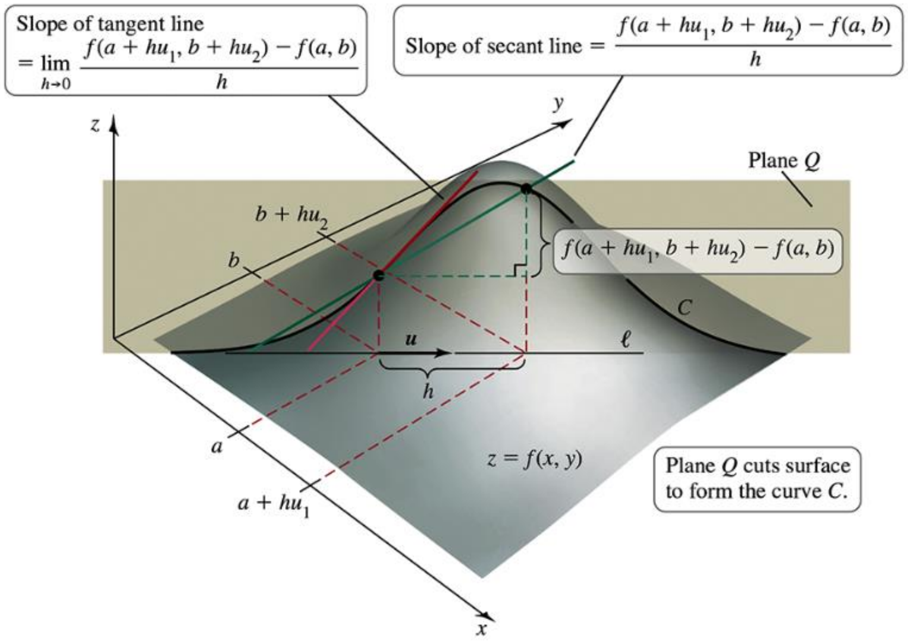
Directional Derivatives

Definition

Let f be differential at (a, b) and let $u = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The directional derivative of f at (a, b) in the direction of u is

$$D_u f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$

provided the limit exists.



Directional Derivatives

Theorem

Let f be differentiable at (a, b) and let $\boldsymbol{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane.

The **directional derivative of $\boldsymbol{f}$ at $(\boldsymbol{a}, \boldsymbol{b})$ in the direction of u** is

$$D_u f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle.$$

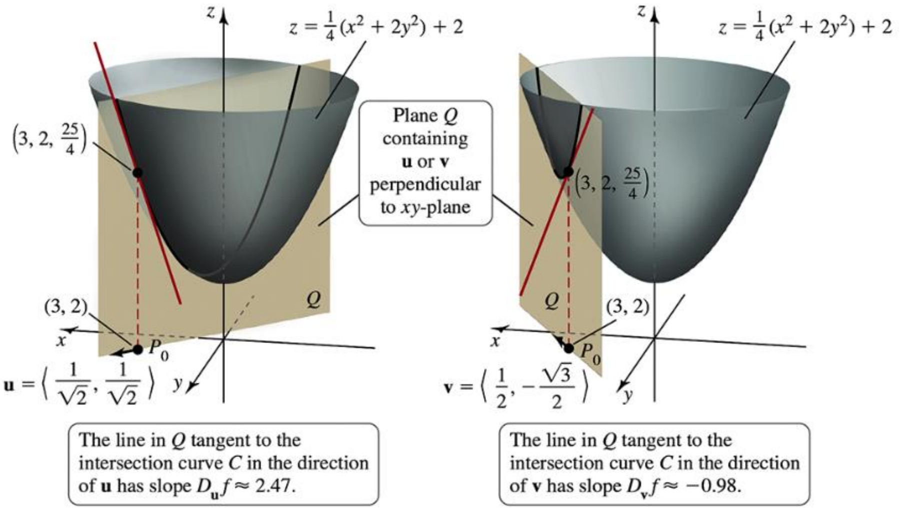
Directional Derivatives

Example

Consider the paraboloid $z = f(x, y) = \frac{1}{4}(x^2 + 2y^2) + 2$.
Let P_0 be the point $(3, 2)$ and consider the unit vectors

$$\boldsymbol{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \text{ and } \boldsymbol{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle.$$

- Find the directional derivative of f at P_0 in the directions of u and v.
- Graph the surface and interpret the directional derivatives.



Gradient Vector

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Definition

Let f be differentiable at the point (x, y). The gradient of f at (x, y) is the vector-valued function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = f_x(x, y)\, \boldsymbol{i} + f_y(x, y)\, \boldsymbol{j}.$$

Gradient Vector

Example

Let $f(x,y) = 3 - \frac{x^2}{10} + \frac{xy^2}{10}$.

a) Compute $\nabla f(3,-1)$.

b) Compute $D_u f(3,-1)$, where $u = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$.

c) Compute the directional derivative of f at $(3,-1)$ in the direction of the vector $\langle 3,4 \rangle$.

Interpretations of the Gradient

Theorem

Let f be differentiable (a, b) with $\nabla f(a, b) \neq 0$.

- f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $|\nabla f(a, b)|$.
- f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-|\nabla f(a, b)|$.
- The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

Interpretations of the Gradient

Example

Consider the bowl-shaped paraboloid $z = f(x, y) = 4 + x^2 + 3y^2$.

a) If you are located on the paraboloid at the point $(2, -\frac{1}{2}, \frac{35}{4})$, in which direction should you move in order to ascend on the surface at the maximum rate? What is the rate of change?

b) If you are located at the point $(2, -\frac{1}{2}, \frac{35}{4})$, in which direction should you move in order to descend on the surface at the maximum rate? What is the rate of change?

c) At the point $(3, 1, 16)$, in what direction(s) is there no change in the function values?

Thank you!