Partial Derivatives

Exercise 1 *Find the first partial derivatives of the following functions.*

a) $f(x,y) = 3x^2y + 2$

b) $f(x,y) = xe^y$

c) $g(x,y) = \cos 2xy$

d) $f(x,y) = e^{x^2y}$

e) $f(w,z) = \dfrac{w}{w^2 + z^2}$

f) $G(s,t) = \dfrac{\sqrt{st}}{s + t}$

Exercise 2 *Find the four second partial derivatives of the following functions.*

a) $h(x,y) = x^3 + xy^2 + 1$

b) $f(x,y) = x^2y^3$

c) $g(x,y) = y^3 \sin 4x$

d) $p(u,v) = \ln(u^2 + v^2 + 4)$

Exercise 3 *Verify that $f_{xy} = f_{yx}$ for the following functions.*

a) $f(x,y) = 3x^2y^{-1} - 2x^{-1}y^2$

b) $f(x,y) = \sqrt{xy}$

Exercise 4 *Find the first partial derivatives of the following functions.*

a) $g(x,y,z) = 2x^2y - 3xz^4 + 10y^2z^2$

b) $F(u,v,w) = \dfrac{u}{v + w}$

c) $G(r,s,t) = \sqrt{rs + rt + st}$

d) $g(w,x,y,z) = \cos(w + x)\sin(y - z)$

Exercise 5 *Find the following derivatives.*

a) $\dfrac{dz}{dt}$, *where* $x^2 + y^3$, $x = t^2$, *and* $y = t$

b) $\dfrac{dz}{dt}$, *where* $z = xy^2$, $x = t^2$, *and* $y = t$

c) $\dfrac{dz}{dt}$, *where* $z = x \sin y$, $x = t^2$, *and* $y = 4t^3$

d) $\dfrac{dz}{dt}$, *where* $z = x^2y - xy^3$, $x = t^2$, *and* $y = t^{-2}$

e) $\dfrac{dz}{dt}$, *where* $z = \sqrt{r^2 + s^2}$, $r = \cos 2t$ *and* $s = \sin 2t$

Exercise 6 *The volume of a right circular cylinder with radius r is and height of h is $V = \pi r^2 h$.*

a) *Assume that r and h are functions of t. Find $V'(t)$.*

b) *Suppose $r = e^t$ and $h = e^{-2t}$, for $t \geq 0$. Use part (a) to find $V'(t)$.*

c) *Does the volume of the cylinder in part (b) increase or decrease as t increases?*

Exercise 7 *Find the following derivatives.*

a) z_s *and* z_t*, where* $z = xy - x^2y$*,* $x = s + t$*, and* $y = s - t$

b) z_s *and* z_t*, where* $z = \sin x \cos 2y$*,* $x = s + t$*, and* $y = s - t$

c) w_s *and* w_t*, where* $\dfrac{x - z}{y + z}$*,* $x = s + t$*,* $y = st$*, and* $z = s - t$

Directional Derivatives and the Gradient

Exercise 8 *Compute the gradient of the following functions and evaluate it at the given point P.*

a) $f(x,y) = 2 + 3x^2 - 5y^2$*;* $P(2, -1)$

b) $g(x,y) = x^2 - 4x^2y - 8xy^2$*;* $P(-1, 2)$

c) $p(x,y) = \sqrt{12 - 4x^2 - y^2}$*;* $P(-1, 1)$

d) $f(x,y) = xe^{2xy}$*;* $P(1,0)$

e) $f(x,y) = \sin(3x + 2y)$*;* $P(\pi, 3\pi/2)$

f) $h(x,y) = \ln(1 + x^2 + 2y^2)$*;* $P(2, -3)$

Exercise 9 *Compute the directional derivative of the following functions at the given point P in the direction of the given vector. Be sure to use a unit vector for the direction vector.*

a) $f(x,y) = x^2 - y^2$*;* $P(-1, -3)$*;* $\left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$

b) $f(x,y) = 10 - 3x^2 + \frac{y^4}{4}$*;* $P(2, -3)$*;* $\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$

c) $g(x,y) = \sin \pi(2x - y)$*;* $P(-1, -1)$*;* $\left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle$

d) $f(x,y) = \sqrt{4 - x^2 - 2y}$*;* $P(2, -2)$*;* $\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$

e) $f(x,y) = 13e^{xy}$*;* $P(1, 0)$*;* $\langle 5, 12 \rangle$

f) $h(x,y) = \ln(4 + x^2 + y^2)$*;* $P(-1, 2)$*;* $\langle 2, 1 \rangle$

Exercise 10 *Consider the following functions and points P.*

- *Find the unit vectors that give the direction of steepest ascent and steepest descent at P.*
- *Find a vector that points in a direction of no change in the function at P.*

a) $f(x,y) = x^2 - 4y^2 - 9$*;* $P(1, -2)$

b) $f(x,y) = x^4 - x^2y + y^2 + 6$; $P(-1,1)$

Exercise 11 *A function* $f(x,y) = 10 - 2x^2 - 3y^2$ *and a point* $P(3,2)$ *are given. Let* θ *correspond to the direction of the directional derivative.*

a) Find the gradient and evaluate it at P.

b) Find the angles θ *(with respect to the positive x-axis) associated with the directions of maximum increase, maximum decrease, and zero change.*

c) Find the angles θ *(with respect to the positive x-axis) associated with the directions of maximum increase, maximum decrease, and zero change.*

d) Find the value of θ *that maximizes* $g(\theta)$ *and find the maximum value.*

e) Verify that the value of θ *that maximizes* g *corresponds to the direction of the gradient. Verify that the maximum value of* g *equals the magnitude of the gradient.*