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Calculus
CSCI 2025
Session 3

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Partial Derivatives

What do we already know?

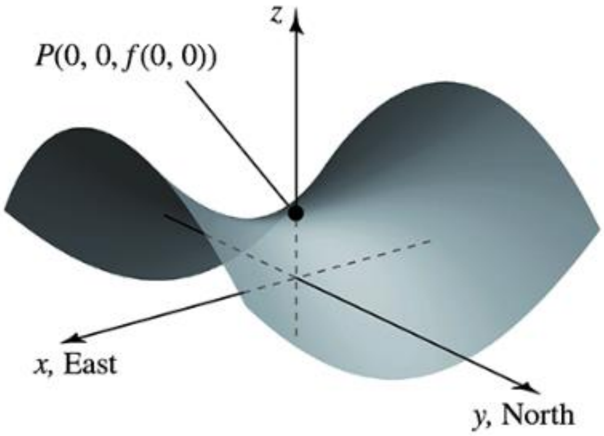
- The derivative of a function of one variable, $y = f(x)$, measures the rate of change of y with respect to x, and it gives slopes of tangent lines.
- Could we use an analogous idea for functions of several variables? For instance, $f(x, y) = x^3 - y^2 + 4$.
- Yes! Derivatives may be defined with respect to any of the independent variables.

Partial Derivatives

Derivatives with two variables

Question: determine the slope of the surface where you are standing!

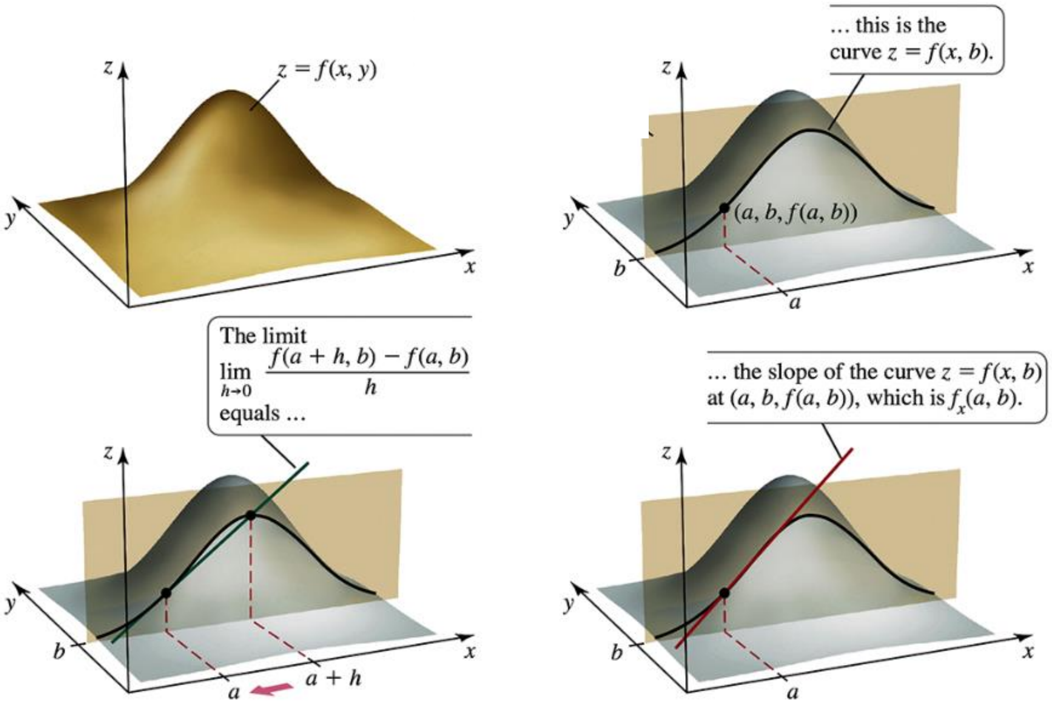
Answer: It depends!



Partial Derivatives

Suppose we move…

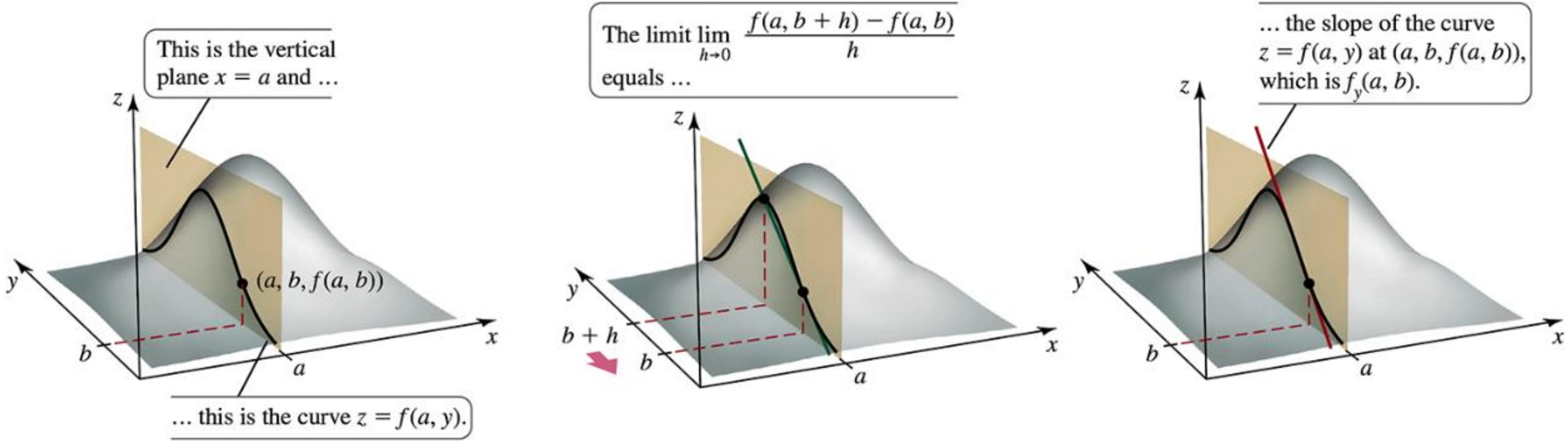
- along the surface $z = f(x, y)$
- starting at $(a, b, f(a, b))$
- in such a way that $y = b$ is fixed and only x varies



Partial Derivatives

Suppose we move…

- along the surface $z = f(x, y)$
- starting at $(a, b, f(a, b))$
- in such a way that $x = a$ is fixed and onlx y varies



Partial Derivatives

Definition

The partial derivative of f with respect to x at the point (a, b) is

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

The partial derivative of f with respect to y at the point (a, b) is

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}.$$

Provided these limits exist.

Partial Derivatives

Notation

The partial derivatives evaluated at a point (a, b) are denoted in any of the following ways:

$$\frac{\partial f}{\partial x}(a, b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a, b)$$

$$\frac{\partial f}{\partial y}(a, b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a, b)$$

Partial Derivatives

Example

Let $f(x, y) = x^3 - y^2 + 4$.

- Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
- Evaluate each derivative at $(2, -4)$.

$\frac{\partial f}{\partial x} = 3x^2 \downarrow 12$, $\frac{\partial f}{\partial y} = -2y \downarrow 8$

Partial Derivatives

Example

Let $f(x, y) = sinxy$. Compute the partial derivatives.

$f_x(x,y) = y \cdot \cos xy$

$f_y(x,y) = x \cdot \cos xy$

Partial Derivatives

Higher Order Partial Derivatives

Given a function f and its partial derivative f_x, we can take the derivative of f_x with respect to x or with respect to y.

This accounts for two of the four possible *second-order partial derivatives.*

Notation 1	Notation 2	What we say…
$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}$	$(f_x)_x = f_{xx}$	d squared f dx squared of $f-x-x$
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}$	$(f_y)_y = f_{yy}$	d squared f dy squared of $f-y-y$
$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x \partial y}$	$(f_y)_x = f_{yx}$	$f-y-x$
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y \partial x}$	$(f_x)_y = f_{xy}$	$f-x-y$

Partial Derivatives

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Example

Find the four second partial derivatives of $f(x, y) = 3x^4y - 2xy + 5xy^3$.

$f_x = 12x^3y - 2y + 5y^3$

$f_y = 3x^4 - 2x + 15xy^2$

$f_{xx} = 36x^2y$

$f_{yy} = 30xy$

$f_{xy} = 12x^3 - 2 + 15y^2$

$f_{yx} = 12x^3 - 2 + 15y^2$

Partial derivatives with more than three variables

Example

Find f_x, f_y, and f_z when $f(x, y, z) = e^{-xy} cosz$.

$f_x = -ye^{-xy} \cos z$

$f_y = -xe^{-xy} \cos z$

$f_z = -e^{-xy} \sin z$

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Thank you!

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