Discrete Mathematics and Algebra
Linear Algebra - Example Population

Arno Kimeswenger

Institute for Mathematics and Statistics
University of Applied Sciences Wiener Neustadt

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Mathematical Model of Population Growth

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Mathematical Model of Population Growth

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- **We discuss discrete model for time steps $0, 1, 2, \ldots$, e.g.**
  $x_2 = 105$ means in year 2 there is about 105 tonnes plant biomass
  $y_3 = 58$ means in year 3 there are 58 individuals in rabbit population,
  $z_7 = 29$ means in year 7 there are 29 individuals in fox population

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- **Plant population $x_0 = 100$ and $x_{t+1} = 1.7x_t - 0.006x_t^2 - 0.001x_t y_t$**
  - **Initial condition $x_0 = 100$ tonnes**
  - **Is increasing due to reproduction $\Rightarrow +1.7x_t$**
  - **Plants fight for space therefore reduction $\Rightarrow -0.006x_t^2$**
    Square makes sense: x_t is small $\Rightarrow$ no real fight but x_t large $\Rightarrow$ fight for space
  - **Plants are food for rabbits therefore reduction $\Rightarrow -0.001x_t y_t$**
    Product makes sense: lots of food x_t and many eaters y_t both amplify reduction

Mathematical Model of Population Growth

- **Rabbit population** $y_0 = 70$ and $y_{t+1} = 0.99y_t + 0.003x_t y_t - 0.012y_t z_z$
  - Initial condition $y_0 = 70$ rabbits
  - Reduction due to natural death $+0.99y_t$ (minus 1%)
  - Lots of food (plants, x_t) increase population $+0.003x_t y_t$
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- **Fox population $z_0 = 30$ and $z_{t+1} = 0.975z_t + 0.0003y_t z_t$**
  - **Initial condition $z_0 = 30$ foxes**
  - **Reduction due to natural death $+0.975z_t$ (minus 2.5%)**
  - **Lots of food (rabbits y_t) increase population $+0.0003y_t z_t$**

Mathematical Model of Population Growth

- **Final non-linear model**

$$x_{t+1} = 1.7x_t - 0.006x_t^2 - 0.001x_t y_t$$
$$y_{t+1} = 0.99y_t + 0.003x_t y_t - 0.012y_t z_z$$
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- **Initial conditions $x_0 = 100$, $y_0 = 70$ and $z_0 = 30$**

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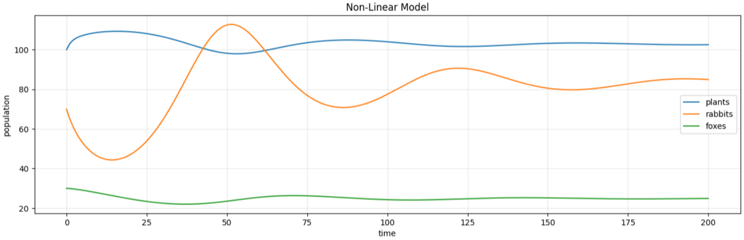

Mathematical Model of Population Growth

- **Fixed point/equilibrium:** no change i.e. find x_t, y_t and z_t such that

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- **More interesting: III: $y_t \approx 83.33 \quad \Rightarrow \quad$ I: $x_t \approx 102.78 \quad \Rightarrow \quad$ II: $z_t = 24.86$**

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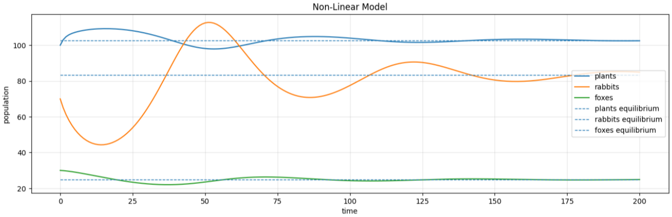

Linear Mathematical Model of Population Growth

- **Non-linear problems are complicated therefore find good linear approximation**
- **Using some techniques (2nd semester, Taylor series) we get approximation**

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} 102.78 \\ 83.33 \\ 24.86 \end{pmatrix} + \begin{pmatrix} 0.38 & -0.10 & 0.00 \\ 0.25 & 1.00 & -1.00 \\ 0.00 & 0.01 & 1.00 \end{pmatrix} \cdot \begin{pmatrix} x_t - 102.78 \\ y_t - 83.33 \\ z_t - 24.86 \end{pmatrix}$$

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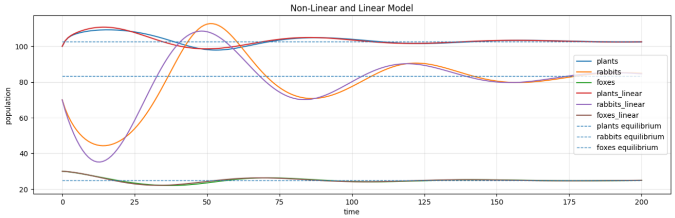
- **Linear model is fine if x_0, y_0 and z_0 close to equilibrium**



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Characterization of Fixed Point

- **Matrix A**

$$A \approx \begin{pmatrix} 0.38 & -0.10 & 0.00 \\ 0.25 & 1.00 & -1.00 \\ 0.00 & 0.01 & 1.00 \end{pmatrix}$$

- **Eigenvalues $\lambda_1 \approx 0.43$, $\lambda_2 \approx 0.98 + 0.09i$ and $\lambda_3 \approx 0.98 - 0.09i$**
- **Absolute values of eigenvalues $|\lambda_1| = 0.43$ and $|\lambda_2| = |\lambda_3| \approx 0.98$**

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  D... diagonal matrix with eigenvalues in diagonal
  T... eigenvectors in columns

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- Diagonalization of matrix: $A = T \cdot D \cdot T^{-1}$
  $D \ldots$ diagonal matrix with eigenvalues in diagonal
  $T \ldots$ eigenvectors in columns
- **All absolute values are less than 1 $\Rightarrow A^n = T \cdot D^n \cdot T^{-1} \xrightarrow{n\to\infty} T \cdot N \cdot T^{-1} = N$**
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  N is null matrix
- **$\vec{v}_n \xrightarrow{n\to\infty} N \cdot \vec{v}_0 = \vec{0}$ and**

$$\vec{v}_n = \begin{pmatrix} x_t - 102.78 \\ y_t - 83.33 \\ z_t - 24.86 \end{pmatrix} \xrightarrow{n\to\infty} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} \xrightarrow{n\to\infty} \begin{pmatrix} 102.78 \\ 83.33 \\ 24.86 \end{pmatrix}$$

Characterization of Fixed Point

- **Complex eigenvalues have polar form, e.g.** $\lambda_2 \approx 0.98 + 0.09i$

$$\Rightarrow \lambda_2 \approx 0.98 \cdot e^{0.08i}$$

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- **To compute $\vec{v}_n$ we need D^n therefore we have to compute e.g. λ_2^n which is**

$$\lambda_2^n \approx 0.98^n \cdot e^{0.08in} = 0.98 \cdot [\cos(0.08n) + i \cdot \sin(0.08n)]$$

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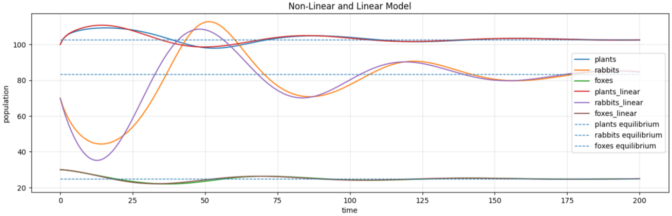
- **This cosine and sine terms explains the oscillating behaviour of solution**



Some Remarks

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  We already knew at the beginning the fixed point

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  We already knew at the beginning the fixed point
- **Yes it is, fixed points can be**
  - **stable/attractive (our case, all eigenvalues' absolute value < 1)**
    If initial condition close to equilibrium $\Rightarrow$ solution convergence to equilibrium
  - **neutrally stable (all eigenvalues' absolute value $= 1$)**
    If initial condition close to equilibrium $\Rightarrow$ solution circles around equilibrium forever
  - **unstable/repelling (at least one eigenvalue's absolute value > 1)**
    Solution diverges in most cases

Unstable Situation

- Our non-linear model again

$$x_{t+1} = 1.7x_t - 0.006x_t^2 - 0.001x_t y_t$$
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- Fixed point: $x_t \approx 102.78$, $y_t \approx 83.33$ and $z_t = 24.86$
- Initial conditions $x_0 = 100$, $y_0 = 70$ and $z_0 = 30$

Unstable Situation

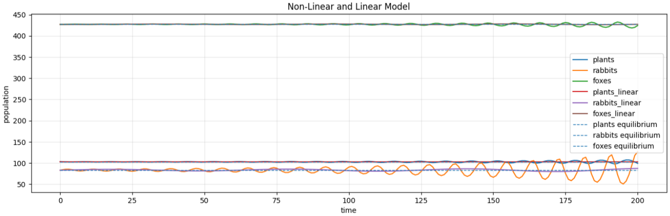
- Changing the parameters a little bit

$$x_{t+1} = 1.7x_t - 0.006x_t^2 - 0.001x_t y_t$$
$$y_{t+1} = 0.99y_t + 0.05x_t y_t - 0.012y_t z_z$$
$$z_{t+1} = 0.975z_t + 0.0003y_t z_t$$

- New fixed point $x_t \approx 102.78$, $y_t \approx 83.33$ and $z_t \approx 427.41$
- New initial conditions $x_0 = 103$, $y_0 = 83$, $z_0 = 427$
  Extremely close to fixed point

Unstable Situation

- In the plot we see the non-linear and the corresponding linear model
- Both models are almost identically



Unstable Situation

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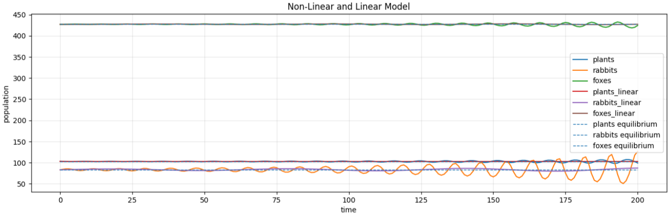
- Both models are almost identically
- **But we see that e.g. rabbit population oscillates with growing magnitued**
- **Eigenvalues: $\lambda_1 \approx 0.035$, $\lambda_2 \approx 0.999 + 0.102i$ and $\lambda_3 \approx 0.999 - 0.102i$**
- **Absolute values of eigenvalues: $|\lambda_1| \approx 0.034$ and $|\lambda_2| = |\lambda_3| \approx 1.004 > 1$**
- **Due to $|\lambda_2| = |\lambda_3| > 1$ we get unstable situation**



Final Model without Interactions

- **Let's see what happens if model is much simpler: no interactions**
  - **Plants are not eaten by rabbits**
  - **Rabbits are not eaten by foxes**

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- **Our non-linear model again (interaction terms in red)**

$$x_{t+1} = 1.7x_t - 0.006x_t^2 - 0.001x_t y_t$$
$$y_{t+1} = 0.99y_t + 0.003x_t y_t - 0.012y_t z_z$$
$$z_{t+1} = 0.975z_t + 0.0003y_t z_t$$

- **Initial conditions $x_0 = 100$, $y_0 = 70$ and $z_0 = 30$**

Final Model without Interactions

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- **Model without interactions**

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Final Model without Interactions

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- Model without interactions

$$
\begin{aligned}
x_{t+1} &= 1.7x_t - 0.006x_t^2 \\
y_{t+1} &= 0.99y_t \\
z_{t+1} &= 0.975z_t
\end{aligned}
$$

- Initial conditions $x_0 = 100$, $y_0 = 70$ and $z_0 = 30$ (no change)
- **New fixed point $x_t \approx 116.67$, $y_t = z_t = 0$**
- **Rabbits and foxes are dying out because $|0.99| < 1$ and $|0.975| < 1$**

Final Model without Interactions

- **Due to quadratic term in**

$$x_{t+1} = 1.7x_t - 0.006x_t^2$$

we get an upper bound for plant population $x_t \leq 116.67$
This is related to the famous logistic growth

Final Model without Interactions

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- **In graph we see non-linear and linear model which are almost identically**