Discrete Mathematics and Algebra
Linear Algebra - Matrices

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Literature

- **Anthony Croft and Robert Davison**
  Mathematics for Engineers
  5$^{\text{th}}$ Edition
  Pearson 2019

Matrices

- **Definition (Matrix, Dimension).** A matrix is a rectangular array of (most) real numbers like

$$A = \begin{pmatrix} 1 & 2 \\ -1.5 & 4 \\ 3 & 2 \\ 7 & 8 \end{pmatrix} \quad \text{or} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ \sqrt{2} & 4 & 5 \\ -3 & 2 & 3 \end{pmatrix}$$

The number of rows and columns defines the dimension (4×2 and 3×3 in the above example) and we write

$$A \in \mathbb{R}^{4\times 2} \quad \text{and} \quad B \in \mathbb{R}^{3\times 3}$$

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- A vector can be seen as a $\mathbb{R}^{n\times 1}$ matrix
- **A matrix can be seen as a mapping/function, e.g. in the above example**

$$A : \{1,2,3,4\} \times \{1,2\} \to \mathbb{R} \quad \text{with} \quad (i,j) \mapsto A(i,j) = a_{i,j}$$

  E.g. $A(1,1) = 1, \quad A(1,2) = 2, \quad A(2,1) = -1.5, \quad \ldots$
  The first index is correlated to the row, the second to the column

Matrices

- **If number of rows is equal to number of columns, the matrix is called quadratic**

- **Examples** $N = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ $\quad D = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{3} & 0 \\ 0 & 0 & \boxed{5} \end{pmatrix}$ $\quad D = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{3} & 0 \\ 0 & 0 & \boxed{0} \end{pmatrix}$ $\quad I = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix}$

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- **Diagonal Matrix** $D \in \mathbb{R}^{m \times m}$ **with** $D(i,j) = \begin{cases} d_{i,i} & \text{if } i = j \\ 0 & \text{else} \end{cases}$ **and** $d_{i,i} \in \mathbb{R}$

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- **Upper Triangular Matrix** $T \in \mathbb{R}^{m\times m}$ **with** $T(i,j) = \begin{cases} t_{i,j} & \text{if } i \leq j \\ 0 & \text{else} \end{cases}$ **and** $t_{i,j} \in \mathbb{R}$
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- **Lower Triangular Matrix $L \in \mathbb{R}^{m \times m}$ with $L(i,j) = \begin{cases} \ell_{i,j} & \text{if } i \leq j \\ 0, & \text{else} \end{cases}$ and $\ell_{i,j} \in \mathbb{R}$**
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Matrix Arithmetic

- **Definition (Addition, Subtraction).** If matrices A and B have the same dimension we can define the addition and subtraction component wise

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 4 & 6 \end{pmatrix}$$

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- Pointwise multiplication/division is not common (although some software tools have them implemented like MATLAB with `.*` or `./` or python with * or /
- **Definition (Multiplication with Scalar). Let $s \in \mathbb{R}$ be a scalar and A be a matrix, then the multiplication $s \cdot A$ is defined by multiplying s to each component of A**

$$5 \cdot \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 5\cdot 2 & 5\cdot 3 \\ 5\cdot 4 & 5\cdot 5 \\ 5\cdot 6 & 5\cdot 7 \end{pmatrix} = \begin{pmatrix} 10 & 15 \\ 20 & 25 \\ 30 & 35 \end{pmatrix}$$

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- **Definition (Matrix-Vector-Multiplication).** If a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $\vec{u} \in \mathbb{R}^{n}$ are given the matrix-vector-multiplication is well defined and the result is a vector $\vec{v} = A \cdot \vec{u} \in \mathbb{R}^{m}$ with

$$\vec{v}(i) = \sum_{k=1}^{n} A(i,k) \cdot \vec{u}(k)$$

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- **Example. Compute the matrix vector multiplication of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$**

  Solution: $\begin{pmatrix} 38 \\ 92 \end{pmatrix}$

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- **Example. You want to buy 2 breads, 3 milks, and 1 rice. Two stores have different prices in Euro (bread, milk, rice): 3.00, 1.20 and 2.50 in shop A and 2.80, 1.50 and 2.30 in shop B. Compute the basket costs with the help of matrix-vector-multiplication.**

  Solution: $\begin{pmatrix} 12.10 \\ 12.4 \end{pmatrix}$

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- **Example.** A matrix is given by $R = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ and a vector by $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$. α is an arbitrary angle. Find out the geometric interpretation (no proof) of the matrix-vector-multiplication. Try e.g. $\alpha = \frac{\pi}{3}\,\mathrm{rad} = 60^\circ$ and $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and compute $R \cdot \vec{u}$ and $R \cdot \left(R \cdot \vec{u}\right)$

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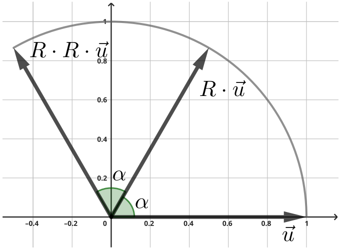
  Solution: $R = \frac{1}{2} \cdot \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$, $R \cdot \vec{u} = \frac{1}{2} \cdot \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$, $R \cdot R \cdot \vec{u} = \frac{1}{4} \cdot \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix}$, rotation by 60° counter-clockwise



Aside - Polar Coordinates

- A vector/point in $\mathbb{R}^2$ can be represented in Cartesian Coordinates with the basis $\left\{\binom{1}{0}, \binom{0}{1}\right\}$

$$\vec{u} = \binom{7}{5} = 7 \cdot \binom{1}{0} + 5 \cdot \binom{0}{1}$$

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- **We have also seen different basis, e.g. $S = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\}$ and therefore**

$$\vec{u} = \begin{pmatrix}7\\5\end{pmatrix} = 3 \cdot \begin{pmatrix}1\\1\end{pmatrix} + 2 \cdot \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}_S$$

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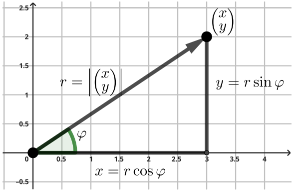
- **Different approach: a vector/point is represented by its radius r and angle φ, because**

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}r\cos\varphi\\r\sin\varphi\end{pmatrix}$$

r and φ are called polar coordinates r and φ are uniquely determined by x and y except for $x = y = 0 \Rightarrow r = 0$ and φ arbitrary



Matrix Arithmetic

Back to rotation matrix:

- After rotation by angle α, a vector $\vec{u} = \binom{r\cos\varphi}{r\sin\varphi}$ is transformed to $\vec{w} = \begin{pmatrix} r\cos(\varphi+\alpha) \\ r\sin(\varphi+\alpha) \end{pmatrix}$

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- **Angle sum identities**

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- **Therefore**

$$\vec{w} = \begin{pmatrix} r\cos\varphi\cos\alpha - r\sin\varphi\sin\alpha \\ r\sin\varphi\cos\alpha + r\cos\varphi\sin\alpha \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}}_{=:R} \cdot \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} = R\cdot\vec{u}$$

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- **We have proven: $R\cdot\vec{u}$ defines a rotation by angle α (conter-clockwise)**

Matrix Arithmetic

- **Definition (Matrix-Matrix-Multiplication).** If two matrices $A \in \mathbb{R}^{\ell \times m}$ and $B \in \mathbb{R}^{m \times n}$ are given, the matrix-matrix-multiplication is well defined and the result is a matrix $C = A \cdot B \in \mathbb{R}^{\ell \times n}$ with

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- **$C(i,j)$ is given by the scalar product of the i^{th} row of A and the j^{th} column of B**

$$\begin{pmatrix} 1 & 2 & 3 \\ \boxed{4} & \boxed{5} & \boxed{6} \end{pmatrix} \cdot \begin{pmatrix} 11 & 12 & \boxed{13} & 14 \\ 15 & 16 & \boxed{17} & 18 \\ 19 & 20 & \boxed{21} & 22 \end{pmatrix} = \begin{pmatrix} 98 & 104 & 110 & 116 \\ 233 & 248 & \boxed{263} & 278 \end{pmatrix}$$

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- **The matrix-matrix-multiplication is an extension of the matrix-vector-multiplication**

Matrix Arithmetic

- **Definition (Transposed Matrix).** For a matrix $A \in \mathbb{R}^{m\times n}$ we define the transposed matrix $B = A^\top \in \mathbb{R}^{n\times m}$ by $B(i,j) = A(j,i)$

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- **A squared matrix $A \in \mathbb{R}^{m \times m}$ is called symmetric if $A^\top = A$**

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^\top = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

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Rules of arithmetic for matrices A and B and scalars λ and μ

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- $\boldsymbol{\lambda \cdot (A + B) = \lambda \cdot A + \lambda \cdot B}$

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- $(A + B)^\top = A^\top + B^\top$
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- $\lambda \cdot (A + B) = \lambda \cdot A + \lambda \cdot B$
- $\left(A^\top\right)^\top = A$

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- $(A + B)^\top = A^\top + B^\top$
- $(\lambda + \mu) \cdot A = \lambda \cdot A + \mu \cdot A$
- $\lambda \cdot (A + B) = \lambda \cdot A + \lambda \cdot B$
- $(A^\top)^\top = A$
- **But $A \cdot B \neq B \cdot A$**

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \\ 23 & 34 \end{pmatrix} \quad \text{but} \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \text{? not defined}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 7 & 7 \end{pmatrix} \quad \text{but} \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 4 & 6 \end{pmatrix}$$

Matrix Arithmetic

Further Rules of arithmetic for matrices A and B and scalars λ and μ

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- $A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k \text{ times}}$

Linear Function

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- **Limits are the key ingredients of analysis (differentiation, integration)**

System of Linear Equations

- **Definition (System of Linear Equations).** A system of linear equations (short linear system) with known matrix $A \in \mathbb{R}^{m\times n}$, known vector $\vec{f} \in \mathbb{R}^m$ but unknown vector $\vec{x} \in \mathbb{R}^n$ is given by

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- **3 equations and 3 unknowns:**

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \Leftrightarrow \quad \begin{array}{rl} \mathrm{I}: & 2x-1y+0z=1 \\ \mathrm{II}: & 1x+2y-2z=2 \\ \mathrm{III}: & 0x-1y+1z=3 \end{array}$$

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- **3 equations and 2 unknowns**

$$\begin{pmatrix} 2 & -1 \\ 1 & 2 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \Leftrightarrow \quad \begin{array}{rl} \text{I}: & 2x-1y=1 \\ \text{II}: & 1x+2y=2 \\ \text{III}: & 0x-1y=3 \end{array}$$

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Elementary row operations

- Multiply an equation with scalar $\lambda \neq 0$ is equivalent to the original equation, e.g. $2 \cdot \mathrm{I}$

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- **Swapping two equations in a linear system is equivalent to the original linear system, e.g.**

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- **Multiply an equation with scalar λ and add the result to another equation is equivalent to the original linear equation, e.g. add $4 \cdot \mathrm{I}$ to II**

$$\begin{array}{rrcl} \mathrm{I}: & 2x-1y+0z & = & 1 \\ \mathrm{II}: & \boxed{1x+2y-2z} & = & \boxed{2} \\ \mathrm{III}: & 0x-1y+1z & = & 3 \end{array} \quad \Leftrightarrow \quad \begin{array}{rrcl} \mathrm{I}: & 2x-1y+0z & = & 1 \\ \mathrm{II}: & \boxed{9x-2y-2z} & = & \boxed{6} \\ \mathrm{III}: & 0x-1y+1z & = & 3 \end{array}$$

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- **Examples of row echelon forms (pivots $\boxed{x}$ and must zeros $\boxed{0}$)**

$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ \boxed{0} & \boxed{4} & 5 \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 2 & 3 \\ \boxed{0} & \boxed{4} & 5 \\ \boxed{0} & \boxed{0} & \boxed{6} \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 0 & 3 \\ \boxed{0} & \boxed{4} & 5 \\ \boxed{0} & \boxed{0} & 0 \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 0 & 3 \\ \boxed{0} & \boxed{0} & \boxed{4} \\ \boxed{0} & \boxed{0} & \boxed{0} \end{pmatrix}$$

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- Solving linear systems in row echelon form is almost trivial
- Transform linear system to row echelon form by using elementary row operations
- **Instead of e.g. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ we use notation $\left(\begin{array}{cc|c} 1 & 2 & f_1 \\ 3 & 4 & f_2 \end{array}\right)$**

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- **Example. Solve**

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Solution: $x = 8$, $y = 15$ and $z = 18$ (unique solution)

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  Solution: $\frac{1}{5} \cdot \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + \frac{s}{5} \cdot \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$ **with** $s \in \mathbb{R}$ **(infinitely many solutions, one degree of freedom dof)**

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$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 8 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$$

  Solution: $\begin{pmatrix} 28 \\ -9 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ **with** $s \in \mathbb{R}$ **(infinitely many solutions, one dof)**

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  Solution: $\begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ **with** $s, t \in \mathbb{R}$ **(infinitely many solutions, two dofs)**

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- **For really large linear systems, e.g. $1\,000\,000$ unknowns we have to use special properties of the matrix if possible (symmetric, positive definite, sparse, etc.)**

System of Linear Equations

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- Gauss elimination is a robust and fast algorithm
- For really large linear systems, e.g. 1 000 000 unknowns we have to use special properties of the matrix if possible (symmetric, positive definite, sparse, etc.)
- **Gauss elimination can also be applied to**
  - **find a basis and the corresponding dimension of a subset of vectors (not here)**
  - **compute the inverse matrix (later)**
  - **compute the determinant (later)**

Inverse Matrix

- **For a scalar $\lambda \neq 0$ there exists another scalar μ such that $\lambda \cdot \mu = \mu \cdot \lambda = 1$**
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- **The inverse matrix is unique: Assume B and C are inverse matrices of A, then there holds**

$$B = B \cdot \underbrace{A \cdot C}_{=I} = \underbrace{B \cdot A}_{=I} \cdot C = C \quad \Rightarrow \quad B = C$$

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Rules of inverse matrix

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- $\left(A^{-1}\right)^{-1} = A$ because $A \cdot A^{-1} = A^{-1} \cdot A = I$
- $\left(A^{\top}\right)^{-1} = \left(A^{-1}\right)^{\top}$ **because** $\left(A^{-1}\right)^{\top} \cdot A^{\top} = \left(A \cdot A^{-1}\right)^{\top} = I^{\top} = I$ **and** $A^{\top} \cdot \left(A^{-1}\right)^{\top} = \left(A^{-1} \cdot A\right)^{\top} = I^{\top} = I$

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- **$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$ because $A \cdot B \cdot B^{-1} \cdot A^{-1} = I$ and $B^{-1} \cdot A^{-1} \cdot A \cdot B = I$**

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- Not trivial: If $A, B \in \mathbb{R}^{m \times m}$ and there holds $A \cdot B = I$ then B is the inverse, i.e. $B = A^{-1}$
- Not trivial: If $A, B \in \mathbb{R}^{m \times m}$ and there holds $B \cdot A = I$ then B is the inverse, i.e. $B = A^{-1}$
- **From the last two statements we conclude that it is sufficient to look for a matrix with $\boldsymbol{A \cdot B = I}$**

Inverse Matrix

- **To illustrate the Gauss elimination for inverse matrix look at an example: For matrix $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ find matrix $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that**

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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- **This is equivalent to two linear systems (with same matrix)**
$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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- **Solve both linear systems simultaneously**
$$\left(\begin{smallmatrix} 1 & 1 & | & 1 \\ 2 & 1 & | & 0 \end{smallmatrix}\right) \quad \textbf{and} \quad \left(\begin{smallmatrix} 1 & 1 & | & 0 \\ 2 & 1 & | & 1 \end{smallmatrix}\right) \quad \textbf{or short} \quad \left(\begin{smallmatrix} 1 & 1 & | & 1 & 0 \\ 2 & 1 & | & 0 & 1 \end{smallmatrix}\right)$$

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- **Use elementary row operations until**
$$\left(\begin{array}{cc|cc} \boxed{1} & \boxed{0} & -1 & 1 \\ \boxed{0} & \boxed{1} & 2 & -1 \end{array}\right) \quad \Leftrightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \quad \Leftrightarrow \quad A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

Inverse Matrix

- **Example.** **Compute the inverse of** $A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix}$

  Solution: $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 2 & 4 \\ -1 & 2 & 5 \end{pmatrix}$

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- **Having the inverse matrix it is possible to solve linear systems (with unique solution)**

$$A \cdot \vec{x} = \vec{f} \quad \Leftrightarrow \quad A^{-1} \cdot f = A^{-1} \cdot A \cdot \vec{x} = I \cdot \vec{x} = \vec{x}$$

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- **Example. Use the matrix from the previous example and solve the linear system** $A \cdot \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

  Solution: $\vec{x} = \begin{pmatrix} 8 \\ 15 \\ 18 \end{pmatrix}$

Inverse Matrix

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Inverse Matrix

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- **Try to find the inverse of** $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$
  Solution: inverse does not exist

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- Try to find the inverse of $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$
  Solution: inverse does not exist
- **For all matrices there exists a pseudo inverse or Moore-Penrose-Inverse which has similar properties as the inverse matrix (not here)**

Determinant

- **Definition (Determinant).** The determinant of a squared matrix $A \in \mathbb{R}^{m \times m}$ is defined as a function $\det\colon \mathbb{R}^{m \times m} \to \mathbb{R}$ and $A \mapsto \det A$ where $\det A$ is defined via:
  - $\det A = A(1,1) \cdot A(2,2) \cdots A(m,m)$, if A is an upper/lower triangular matrix
  - If B is the same matrix as A but two rows are exchanged then $\det B = -\det A$
  - If B is the same matrix as A but one row is multiplied by a scalar λ then $\det B = \lambda \cdot \det A$
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- **Appling the rules we get e.g.**

$$\det\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{5} & 6 \\ 0 & 0 & \boxed{9} \end{pmatrix} = 1\cdot 5\cdot 9 = 45$$

$$\det\begin{pmatrix} \boxed{1} & \boxed{2} & \boxed{3} \\ 4 & 5 & 6 \\ \boxed{7} & \boxed{8} & \boxed{9} \end{pmatrix} = -\det\begin{pmatrix} \boxed{7} & \boxed{8} & \boxed{9} \\ 4 & 5 & 6 \\ \boxed{1} & \boxed{2} & \boxed{3} \end{pmatrix}$$

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$$\det\begin{pmatrix} \boxed{3} & \boxed{6} & \boxed{9} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \det\begin{pmatrix} \boxed{3\cdot 1} & \boxed{3\cdot 2} & \boxed{3\cdot 3} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 3 \cdot \det\begin{pmatrix} \boxed{1} & \boxed{2} & \boxed{3} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\det\begin{pmatrix} \boxed{1} & \boxed{2} & \boxed{3} \\ \boxed{4} & \boxed{5} & \boxed{6} \\ 7 & 8 & 9 \end{pmatrix} = \det\begin{pmatrix} \boxed{1} & \boxed{2} & \boxed{3} \\ \boxed{4+1} & \boxed{5+2} & \boxed{6+3} \\ 7 & 8 & 9 \end{pmatrix} = \det\begin{pmatrix} \boxed{1} & \boxed{2} & \boxed{3} \\ \boxed{5} & \boxed{7} & \boxed{9} \\ 7 & 8 & 9 \end{pmatrix}$$

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- Idea to compute the determinant: use elementary row operations to get upper triangular matrix and then compute the product of the diagonal
- **Example. Compute the determinant of**

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 9 \end{pmatrix}$$

Solution: $0, 0, -72$

Determinant

- **A faster way for 2×2 matrices**

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

because if $a \neq 0$:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{pmatrix} = a \cdot \left(d - \frac{bc}{a} \right) = ad - bc$$

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and if $a = 0$:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = -\det \begin{pmatrix} c & d \\ 0 & b \end{pmatrix} = -bc$$

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- **Example. Compute** $\det \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{pmatrix}$
  Solution: $-\mathbf{2}$

Determinant

- A faster way for 2×2 matrices

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

- Example. Compute $\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

  Solution: -2

- **Example. Show that the inverse of a 2×2 matrix is given by**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^{-1}$$

Determinant

- **A faster way for 3×3 matrices - Rule of Sarrus**

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - gec - hfa - idb$$

$$\begin{array}{|ccc|cc}
+ & + & + & & \\
a & b & c & a & b \\
d & e & f & d & e \\
g & h & i & g & h \\
- & - & - & &
\end{array}$$

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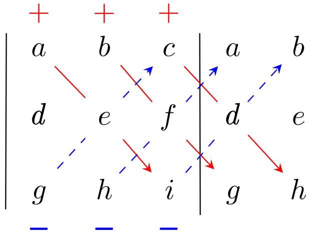

- **Example. Compute the determinant of $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$**

  Solution: 8

Determinant

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- For $m \times m$ matrices there is also the Laplace expansion (not here)
- Using our approach with elementary row operations is useful for larger matrices too
- But in most cases determinants are avoided because they are computational expensive
- Determinants are used to compute eigenvalues, see next topic
- **Rules of determinant**
  - **$\det A \neq 0$ is equivalent to A is regular !!!**
  - **$\det(A \cdot B) = \det A \cdot \det B$**
  - **$\det\left(A^{-1}\right) = \frac{1}{\det A}$**
  - **$\det\left(A^{\top}\right) = \det A$**
  - **If we can get a zero row by using elementary row operations we can use:**
    “If B is the same matrix as A but one row is multiplied by a scalar λ then $\det B = \lambda \cdot \det A$”
    and therefore for $\lambda = 0$ we get $\det A = 0 \cdot \det A = 0$

Eigenvalues

- **Definition (Eigenvalue, Eigenvector).** For $A \in \mathbb{R}^{m \times m}$, $\lambda \in \mathbb{C}$ is called eigenvalue with corresponding eigenvector $\vec{v} \in \mathbb{C}^m$ if $\vec{v} \neq \vec{0}$ and

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- **Some applications of eigenvalues to see the importance:**
  - **Optimization in higher dimensions: Eigenvalues show whether a critical point of a function is a local minimum, maximum, or a saddle point.**
  - **Data Science: The so called principal component analysis (PCA) can be used to reduce dimensionality without loosing much information of a dataset. Eigenvectors correspond to direction and eigenvalues to its variance/importance**
  - **Resonance: Eigenvalues describe the natural frequencies of vibrating objects like guitars, bridges, or skyscrapers. E.g. the collapse of the Tacoma Narrows Bridge, see `https://en.wikipedia.org/wiki/Tacoma_Narrows_Bridge_(1940)`**
  - **Quantum mechanics: The eigenvalues of the Schrödinger operator correspond to the possible energy levels of an electron in a hydrogen atom, and the eigenfunctions correspond to the probability distribution of the electron's position.**

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$$A \cdot \vec{v} = \lambda \cdot \vec{v}$$

- **Equivalent to $\lambda \in \mathbb{C}$, $\vec{v} \neq \vec{0}$ and $A \cdot \vec{v} - \lambda \cdot \vec{v} = \vec{0}$**

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- **$\vec{v} \neq 0$, therefore this is equivalent to $\lambda \in \mathbb{C}$ and $\det(A - \lambda I) = 0$ and then if λ is known compute $\vec{v} \neq 0$ by linear system $(A - \lambda I) \cdot \vec{v} = \vec{0}$**

Eigenvalues

- **Example.** **Find all eigenvalues and corresponding eigenvectors of** $\begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{pmatrix}$

  Solution: $\lambda_1 = 0.5, \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \lambda_2 = 0, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \lambda_3 = 1, \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

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- **Example. Find all eigenvalues and corresponding eigenvectors of** $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

  Solution: $\lambda_1 = 1, \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \lambda_2 = \lambda_3 = 3, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, **no** $\vec{v}_3$

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- **Example. Find all eigenvalues and corresponding eigenvectors of $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$**

  Solution: $\lambda_1 = 1 + i, \vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \lambda_2 = \bar{\lambda}_1 = 1 - i, \vec{v}_2 = \bar{\vec{v}}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

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- **If A is regular and if λ is an eigenvalue of A then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}, because**

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  Eigenvectors are the same
- **Eigenvalues of A and $A^\top$ are the same because we have to solve**

$$\det\left(A^\top - \lambda I\right) = \det\left((A - \lambda I)^\top\right) = \det\left(A - \lambda I\right)$$

  But eigenvectors are in general different

Eigenvalues

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- Product of all eigenvalues is equal to determinant of A
- $\vec{v} = \vec{0}$ is never an eigenvector!!!
- **A symmetric matrix $A \in \mathbb{R}^{m \times m}$ has real eigenvalues and real eigenvectors. Furthermore there are m orthogonal eigenvectors (we like that)**

Eigenvalues

- **Theorem.** If a matrix $A \in \mathbb{R}^{m \times m}$ with eigenvalues $\lambda_1, \ldots, \lambda_m$ and m linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_m$, then there exists a **diagonalization** of A, i.e. the A can be written as

$$A = T \cdot D \cdot T^{-1}$$

where D is the diagonal matrix with eigenvalues in the diagonal
and $T = \left(\vec{v}_1, \ldots, \vec{v}_m\right)$
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- **This can be applied to symmetric matrices, where the eigenvectors are orthogonal $\Rightarrow$ linearly independent**
  Furthermore: If we set length of eigenvectors to 1 we get $T^{-1} = T^{\top}$ which is faster

Eigenvalues

- **Example.** For our first eigenvalue example: Find D, T and T^{-1} and check that there holds $A = T \cdot D \cdot T^{-1}$.
  Next set the length of all eigenvectors to 1 and show that the corresponding matrix T satisfies $A = T \cdot D \cdot T^{\top}$.
  Maybe you want to use Python.

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- **Use diagonalization to compute powers of A, i.e. A^n**

$$A^2 = \underbrace{\left(T \cdot D \cdot T^{-1}\right)}_{=A} \cdot \underbrace{\left(T \cdot D \cdot T^{-1}\right)}_{=A} = T \cdot D \cdot \underbrace{T^{-1} \cdot T}_{=I} \cdot D \cdot T^{-1} = T \cdot D \cdot I \cdot D \cdot T^{-1}$$

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- **And analogously: $\boldsymbol{A^n = T \cdot D^n \cdot T^{-1}}$**

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- The computation of D^n is trivial, e.g.

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^{10} = \begin{pmatrix} 1^{10} & 0 \\ 0 & 2^{10} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\,024 \end{pmatrix}$$

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- **Example. Use the setting of the previous example and compute A^{10} and A^n for $n \to \infty$**

  Solution: $\begin{pmatrix} 0.00097656 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{pmatrix}$ **and** $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{pmatrix}$

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- **Example. Population of 3 species (plants, rabbits, foxes), see slides "Population"**

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- **Use power iteration, QR-algorithm or other methods for real problems (implemented in python - numpy.linalg.eig)**