Discrete Mathematics and Algebra
Linear Algebra - Vectors

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Literature

- **Anthony Croft and Robert Davison**
  Mathematics for Engineers
  5^{th} Edition
  Pearson 2019

Vectors

- **Definition (Vector, Dimension).** A vector is a list of (most) real numbers like

$$\vec{u} = \begin{pmatrix} 1 \\ -1.5 \\ 3 \\ 7 \end{pmatrix} \quad \text{or} \quad \vec{v} = \begin{pmatrix} 1 \\ \sqrt{2} \\ -3 \end{pmatrix}$$

The number of entries defines the dimension (4 and 3 in the above example) and we write

$$\vec{u} \in \mathbb{R}^4 \quad \text{and} \quad \vec{v} \in \mathbb{R}^3$$

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- **Visualization of 2 and 3 dimensional vectors e.g. $\vec{w} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$:**
  From start position P (point) e.g. $P = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ go 4 units in x-direction and 1 unit in y-direction.
  $Q = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ is end position (point)

Vectors

- **Example.** If the start position is given by $P = \binom{3}{1}$ and end position by $Q = \binom{5}{2}$ find the corresponding vector $\vec{u}$. Visualize the situation.
  Solution: $\binom{2}{1}$

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- **In physics there are scalar variables (one number) like mass $80\,\mathrm{kg}$, volume $5\,\mathrm{L}$, work $3\,\mathrm{J}$, power $2\,\mathrm{W}$, temperature $20\,^{\circ}\mathrm{C}$, etc.**

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- In physics there are scalar variables (one number) like mass 80 kg, volume 5 L, work 3 J, power 2 W, temperature 20 °C, etc.
- **But there are also vector valued variables (more than one number, here 2) like position $\binom{2}{4}$ m, velocity $\binom{2}{4}$ m s^{-1}, acceleration $\binom{1}{3}$ m s^{-2}, force $\binom{5}{6}$ kg m s^{-2} = $\binom{5}{6}$ N, etc.**

Vector Arithmetic

- **Definition (Addition, Subtraction).** If vectors $\vec{u}$ and $\vec{v}$ have same dimension we can define the addition and subtraction component wise, e.g.

$$\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}4\\5\\6\end{pmatrix}=\begin{pmatrix}1+4\\2+5\\3+6\end{pmatrix}=\begin{pmatrix}5\\7\\9\end{pmatrix}$$

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- **Pointwise multiplication/division is not common (although some software tools have them implemented like MATLAB with `.*` or `./`**
- **Definition (Multiplication with Scalar). Let s be a scalar (i.e. $s \in \mathbb{R}$) and $\vec{u}$ be a vector, then the multiplication $s \cdot \vec{u}$ is defined by multiplying s to each component of $\vec{u}$, e.g.**

$$5 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \cdot 1 \\ 5 \cdot 2 \\ 5 \cdot 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 15 \end{pmatrix}$$

Vector Arithmetic

- **Example.** The three forces $\vec{F}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{F}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{F}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ act on an object. Which additional force is necessary to neutralize this forces?
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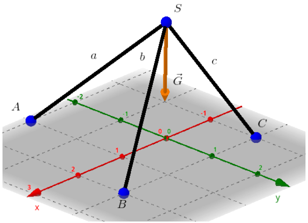
- **Example. A tripod is given by the 3 legs a, b and c. At joint S, a weight force $\vec{G}$ is applied. Which reaction forces occur in the legs?**



$$A = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, B = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, S = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \vec{G} = \begin{pmatrix} 0 \\ 0 \\ -18 \end{pmatrix}$$

Units are m and kN

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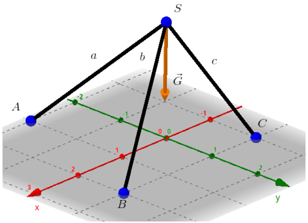

Solution: $\vec{F}_a = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$, $\vec{F}_b = \begin{pmatrix} 5 \\ -5 \\ 10 \end{pmatrix}$ **and** $\vec{F}_c = \begin{pmatrix} -3 \\ 6 \\ 6 \end{pmatrix}$

Length/norm/magnitude of a vector

- **Definition (Length/norm/magnitude of a vector).** For a vector $\vec{v} \in \mathbb{R}^n$ with elements $v_1, v_2, \ldots, v_n$ the length of a vector is given by

$$|\vec{v}| := \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

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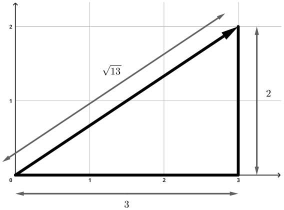
- **Example. Compute the length for $\vec{v} = \binom{3}{2}$ and show that this is exactly the length of the hypotenuse of the corresponding right-angled triangle (Pythagoras).**

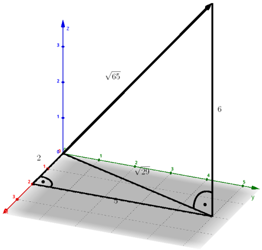
Solution: $\sqrt{13}$



Length/norm/magnitude of a vector

- **Example.** **Compute the length for $\vec{v} = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$ and compare with Pythagoras' theorem.**

  Solution: $\sqrt{65}$



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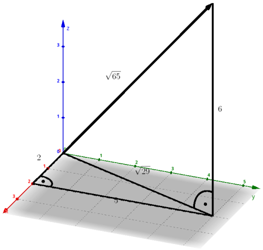

- **Example. Show that $|\lambda \cdot \vec{v}| = |\lambda| \cdot |\vec{v}|$ for $\lambda \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n$. Next use $\lambda = -\frac{1}{3}$ and $\vec{v}$ from above to verify the statement.**

  Solution: $\frac{\sqrt{65}}{3}$

Unit Vector

- **Definition (Unit Vector).** For a vector $\vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$ we define the unit vector

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- **We see that a vector $\vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq 0$ can be rewritten as**

$$\vec{v} = |\vec{v}| \cdot \vec{v}_0$$

  $|\vec{v}|$ defines the magnitude and $\vec{v}_0$ the direction of the vector $\vec{v}$
- **Often we only use the magnitude e.g. for the velocity of a bike we say $20\,\mathrm{km\,h^{-1}}$ and are less interested in the direction but the velocity is still a vector**

Unit Vector

- **Example.** Compute the unit vector of $\vec{v} = \begin{pmatrix} 7/3 \\ 2/3 \\ 5/3 \end{pmatrix}$

  Solution: $\frac{\sqrt{78}}{78} \cdot \begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix}$

Scalar Product

- **Definition (Scalar Product).** For $\vec{u} \in \mathbb{R}^n$ with $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ and $\vec{v} \in \mathbb{R}^n$ with $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ (same dimension!) we define the scalar product by

$$\vec{u} \cdot \vec{v} := \langle \vec{u}, \vec{v} \rangle := u_1 \cdot v_1 + \ldots + u_n \cdot v_n$$

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- **For $\alpha, \beta \in \mathbb{R}$ and $\vec{v}, \vec{w} \in \mathbb{R}^n$ we can skip "·" in the following**

$$\alpha \cdot \beta = \alpha\beta, \quad \alpha \cdot \vec{v} = \alpha\vec{v}$$

  but we never skip "·" for the scalar product $\vec{u} \cdot \vec{v}$
- **The result of the scalar product is a scalar**

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- **Example. Show that $\vec{v} \cdot \vec{v} = |\vec{v}|^2$ or equivalently $\sqrt{\vec{v} \cdot \vec{v}} = |\vec{v}|$. Use the first vector of the previous example to verify this formula.**

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- **We know that for $\alpha, \beta \in \mathbb{R}$ there holds $\alpha \cdot \beta = 0$ if and only if $\alpha = 0$ or $\beta = 0$**
- **Example. Compute the scalar product of the vectors $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Visualize the vectors and try to find an explanation.**

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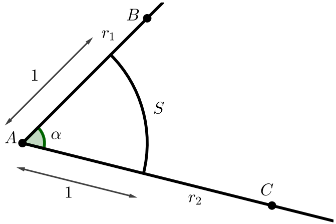
  Solution: 0
- **Definition (normal/orthogonal). Two vectors $\vec{u}$ and $\vec{v}$ are called orthogonal if $\vec{u} \cdot \vec{v} = 0$**

Scalar Product

- **Definition (Angle).** Let A, B and C be three points in $\mathbb{R}^2$. Consider ray r_1 which includes A and B and ray r_2 which includes A and C. for both rays A is the initial point.



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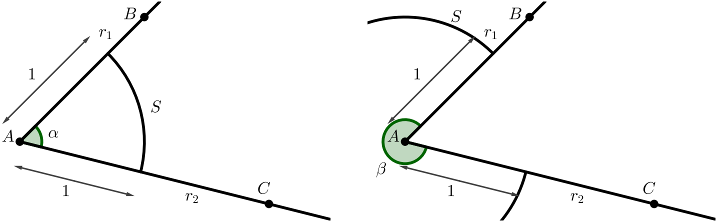

Then the angle α between r_1 and r_2 is defined by the arc length S. The arc length is the part of the circle with centre A and radius 1 which is between the two rays.

- **There is also a second angle β which is defined by using the opposite arc length**

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- **Sometimes we use degree instead of radian**

$$\alpha \text{ rad} = \left(\frac{\alpha \cdot 360}{2\pi}\right)^{\circ}$$

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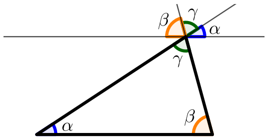
- **For a triangle with angles α, β and γ there holds $\alpha + \beta + \gamma = \pi \text{ rad} = 180^{\circ}$**



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- **Definition (Right Angle).** **An angle of** $\frac{\pi}{2}\,\mathrm{rad} = 90^\circ$ **is called right angle**

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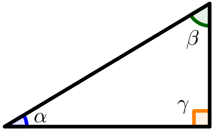
- **Conclusion: For a right angled triangle there holds $\alpha + \beta = \frac{\pi}{2}$ rad $= 90°$**



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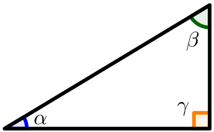
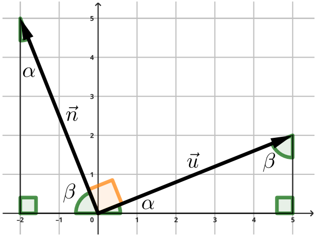
- Conclusion: For a right angled triangle there holds $\alpha + \beta = \frac{\pi}{2}$ rad $= 90°$
- **$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\vec{n} = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$ are orthogonal, i.e. $\vec{u} \cdot \vec{n} = 0$ and the angle between $\vec{u}$ and $\vec{v}$ is a right angle.**





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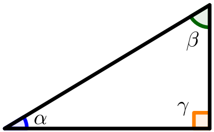
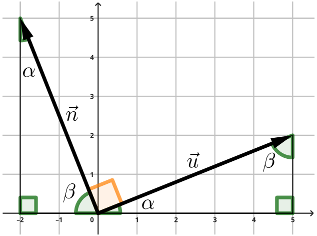
- Conclusion: For a right angled triangle there holds $\alpha + \beta = \frac{\pi}{2}$ rad $= 90°$
- $\vec{u} = \binom{u_1}{u_2}$ and $\vec{n} = \binom{-u_2}{u_1}$ are orthogonal, i.e. $\vec{u} \cdot \vec{n} = 0$ and the angle between $\vec{u}$ and $\vec{v}$ is a right angle.
- **Later we will see that $\vec{u} \cdot \vec{v} = 0$ is equivalent to angle between $\vec{u}$ and$\vec{v}$ is a right angle**





Scalar Product

- If the first ray is defined by the vector $\vec{u} = \overrightarrow{AB}$ (means initial point A and endpoint B) and the second ray by $\vec{v} = \overrightarrow{AC}$ there holds

$$\alpha = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}\right)$$

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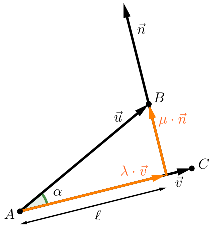
- **Proof.**



For $\vec{u} = \binom{u_1}{u_2}$ define $\vec{n} = \binom{-u2}{u_1}$ $\Rightarrow$ $\vec{u} \cdot \vec{n} = 0$
We know that $\ell = \cos(\alpha) \cdot |\vec{u}|$
Further there holds $\ell = |\lambda \cdot \vec{v}| = |\lambda| \cdot |\vec{v}|$ with $\lambda \in \mathbb{R}$ unknown
Decompose $\vec{u}$ via: $\vec{u} = \lambda \cdot \vec{v} + \mu \cdot \vec{n}$
Also $\mu \in \mathbb{R}$ is unknown (not needed)
$\vec{u} \cdot \vec{v} = \lambda \cdot \vec{v} \cdot \vec{v} + \mu \cdot \vec{n} \cdot \vec{v} = \lambda \cdot |\vec{v}|^2$ $\Rightarrow$ $\lambda = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}$
$\Rightarrow$ $\cos(\alpha) \cdot |\vec{u}| = \ell = |\lambda| \cdot |\vec{v}| = \frac{|\vec{u} \cdot \vec{v}|}{|\vec{v}|}$

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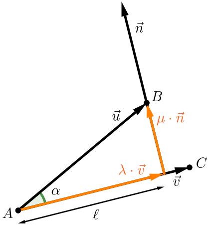
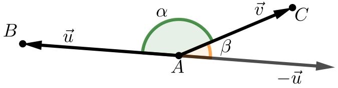


$$\Rightarrow \quad \cos(\alpha)\cdot|\vec{u}| = \ell = |\lambda|\cdot|\vec{v}| = \frac{|\vec{u}\cdot\vec{v}|}{|\vec{v}|}$$

$$\Rightarrow \quad \alpha = \arccos\left(\frac{|\vec{u}\cdot\vec{v}|}{|\vec{u}|\cdot|\vec{v}|}\right)$$

What if $\vec{u}\cdot\vec{v} < 0$? Then $|\vec{u}\cdot\vec{v}| = -\vec{u}\cdot\vec{v} = (-\vec{u})\cdot\vec{v}$

$\Rightarrow$ **sign of one vector changed $\Rightarrow$ supplementary angle $\pi - \alpha$**



Scalar Product

- **Again**

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- **Formula with $\left|\vec{u}\cdot\vec{v}\right|$ results always in acute angle i.e. $0 \leq \alpha \leq \frac{\pi}{2}\,\text{rad} = 90^\circ$**
  So we get supplementary angle if the real angle is larger than $\frac{\pi}{2}\,\text{rad} = 90^\circ$

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  So we get supplementary angle if the real angle is larger than $\frac{\pi}{2}$ rad $= 90°$
- **Formula without absolute value $\vec{u} \cdot \vec{v}$ results in real angle i.e. $0 \leq \alpha \leq \pi$ rad $= 180°$**
  So this angle can be acute or obtuse

Scalar Product

- Again

$$\alpha = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}\right)$$

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- **One can show (Cauchy Schwarz Inequality) that this formula is well defined (if none of the vectors is a $\vec{0}$ vector) because $-1 \le \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|} \le 1$**

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  Because $\arccos(0) = \frac{\pi}{2}$

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- **Final note: We can use this formula to define angles. Especially for higher dimensions or for different vector spaces (whatever this is)**

Scalar Product

- **Example.** **Compute the angle of the vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$**
  Solution: $\frac{\pi}{4}$ rad $=45°$

Scalar Product

- Example. Compute the angle of the vectors $\binom{1}{0}$ and $\binom{1}{1}$

  Solution: $\frac{\pi}{4}$ rad $= 45°$

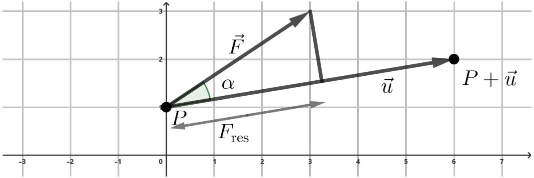
- **Example. Compute the angle of the vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$**

  Solution: $\frac{\pi}{3}$ rad $= 60°$

Scalar Product

- When a constant force $\vec{F}$ and a displacement $\vec{u}$ are given then the work done is given by

$$W = F_{\text{res}} \cdot |\vec{u}|$$



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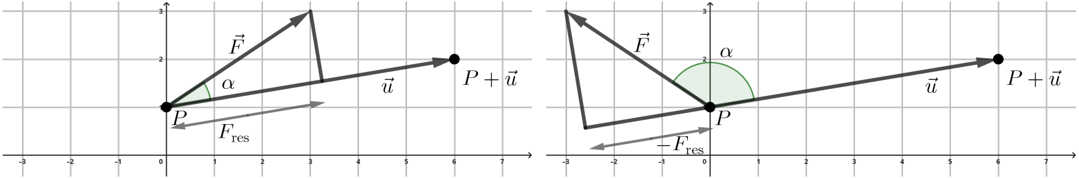

- **If angle $\alpha > \frac{\pi}{2}$ rad $= 90°$ formula is still valid but F_{res} and therefore work W are negative**

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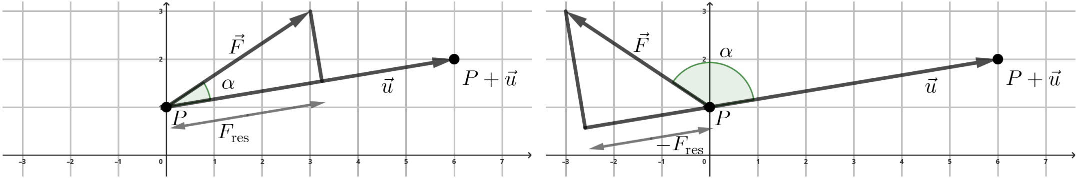

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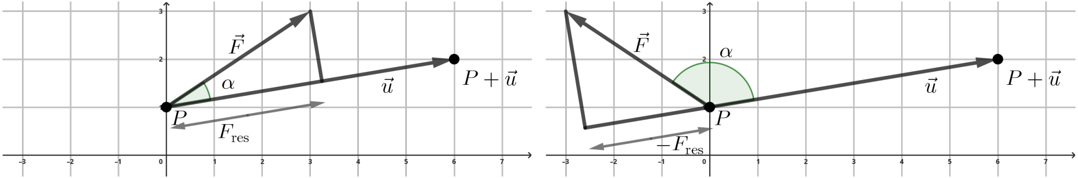

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- $F_{\text{res}} = \left|\vec{F}\right| \cdot \cos(\alpha) = \left|\vec{F}\right| \cdot \frac{\vec{F} \cdot \vec{u}}{\left|\vec{F}\right| \cdot |\vec{u}|} = \frac{\vec{F} \cdot \vec{u}}{|\vec{u}|}$

Scalar Product

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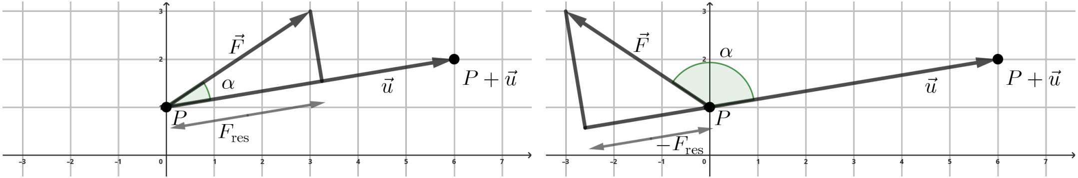

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- **Formula for work:** $W = F_{\text{res}} \cdot \left|\vec{u}\right| = \frac{\vec{F} \cdot \vec{u}}{\left|\vec{u}\right|} \cdot \left|\vec{u}\right| = \vec{F} \cdot \vec{u}$

Scalar Product

- **Example.** **Compute work done if**
  Force $\vec{F} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ N, **start point is** $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ m **and end point is** $\begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$ m.
  Solution: 24J

Cross Product

- **In $\mathbb{R}^2$ we have seen, that we can find orthogonal vector of $\vec{u} = \binom{u_1}{u_2}$ via e.g. $\binom{-u_2}{u_1}$**

Cross Product

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- **In $\mathbb{R}^3$ we have a more complex problem: find $\vec{n} \in \mathbb{R}^3$ which is orthogonal to two vectors $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$**

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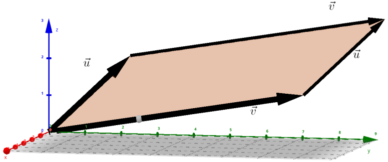
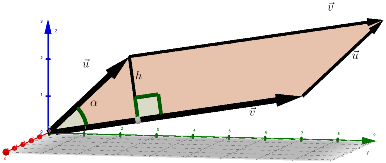
- **Solution is not unique because if $\vec{n} \in \mathbb{R}^3$ is orthogonal to $\vec{u}$ and $\vec{v}$, then e.g. $2 \cdot \vec{n}$ is too $\Rightarrow$ fix length by $|\vec{n}| = A$, where A is the area of the parallelogram**



Cross Product

- **Find formula for Area A**



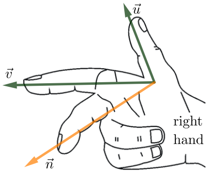
$$A = |\vec{v}| \cdot h = |\vec{v}| \cdot |\vec{u}| \cdot \sin(\alpha) = |\vec{v}| \cdot |\vec{u}| \cdot \sqrt{1 - \cos^2(\alpha)}$$

because $\sin^2(\alpha) + \cos^2(\alpha) = 1$ (Pythagoras) and therefore

$$A = |\vec{v}| \cdot |\vec{u}| \cdot \sqrt{1 - \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}\right)^2} = \sqrt{|\vec{v}|^2 \cdot |\vec{u}|^2 - (\vec{u} \cdot \vec{v})^2}$$

Cross Product

- **Solution is still not unique because if $\vec{n} \in \mathbb{R}^3$ is orthogonal to $\vec{u}$ and $\vec{v}$, then $-\vec{n}$ is too**
  $\Rightarrow$ Fix sign such that $\vec{u}, \vec{v}$ and $\vec{n}$ build a right-handed system



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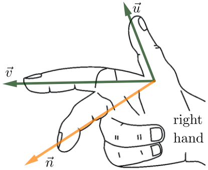

- **Now the solution is fixed by the following cross product**
- **Definition (Cross Product). For $\vec{u}, \vec{v} \in \mathbb{R}^3$ the cross product is divined by**

$$\vec{u} \times \vec{v} := \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} := \begin{pmatrix} u_2 \cdot v_3 - u_3 \cdot v_2 \\ -u_1 \cdot v_3 + u_3 \cdot v_1 \\ u_1 \cdot v_2 - u_2 \cdot v_1 \end{pmatrix}$$

Cross Product

- **Example.** Check that $\vec{u} \cdot \vec{n} = \vec{v} \cdot \vec{n} = 0$

Cross Product

- Example. Check that $\vec{u} \cdot \vec{n} = \vec{v} \cdot \vec{n} = 0$
- **Example. For $\vec{u} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.**
  - **Compute the area A of the corresponding parallelogram by using school maths formula $A = \sin(\alpha) \cdot |\vec{u}| \cdot |\vec{v}|$**
  - **Use the formula $A = \sqrt{|\vec{v}|^2 \cdot |\vec{u}|^2 - (\vec{u} \cdot \vec{v})^2}$**
  - **Compute the cross product $\vec{n}$ and compute $A = |\vec{n}|$**
  - **Check that $\vec{u}, \vec{v}$ and $\vec{n}$ build a right-handed system**

  Solution: $A = 5, \vec{n} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$

Cross Product

- **Attention**

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$
$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

Cross Product

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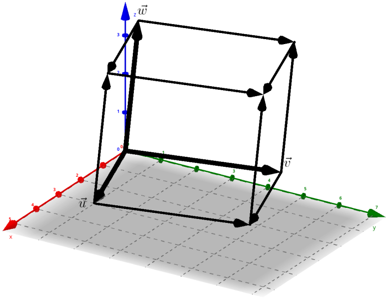
- **In mechanics the cross product of $\vec{r}$ and $\vec{F}$ is the torque $\vec{\tau}$. Here $\vec{r}$ is the position vector and $\vec{F}$ is the force vector, i.e. $\vec{\tau} = \vec{r} \times \vec{F}$**

Cross Product

- **The volume V of the parallelepiped with vectors $\vec{u}, \vec{v}$ and $\vec{w}$ is given by the scalar triple product (see exercises)**

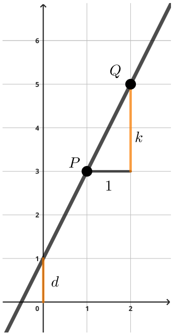


$$V = \left|\left(\vec{u} \times \vec{v}\right) \cdot \vec{w}\right|$$

Lines in Parameter Form

- In $\mathbb{R}^2$ two points $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ and $Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ determine a line

$$ax + by = c$$



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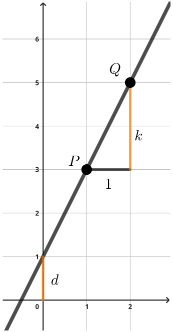
- **If $b \neq 0$ we get the more familiar form with slope k and Intercept d**

$$y = \underbrace{-\frac{a}{b}}_{=k} \cdot x + \underbrace{\frac{c}{b}}_{=d} = k \cdot x + d$$



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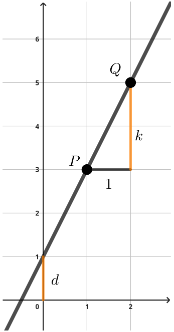

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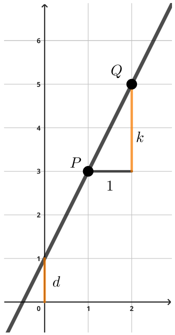


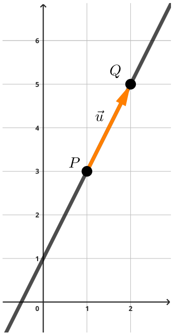
- Line means $x + 1 \Rightarrow y + k$
- **If $b = 0$ it is even simpler because then we the vertical line $x = \frac{c}{a}$**

Lines in Parameter Form

- **This line can also be represented by a set of points which satisfy**

$$\vec{x} = \binom{x}{y} = P + s \cdot \vec{u}$$

where $\vec{u} = \overrightarrow{PQ}$ **and** $s \in \mathbb{R}$



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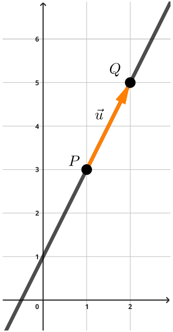

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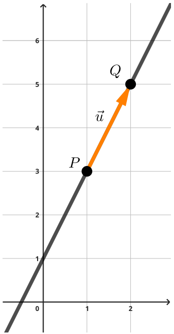
- **We can use this parametric form to define lines in higher dimensions too**



Lines in Parameter Form

- **Example.** **Find the parametric form of the line with $P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$. Find the slope-intercept form too.**
  Solution: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $y = -\frac{4}{3} \cdot x + \frac{7}{3}$

Lines in Parameter Form

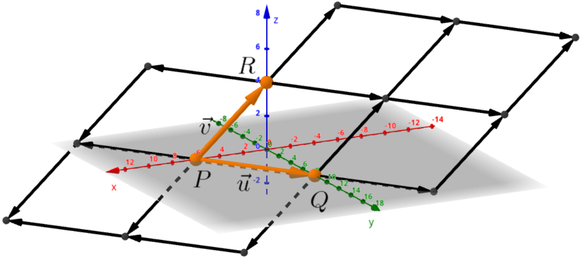
- Example. Find the parametric form of the line with $P = \binom{1}{1}$ and $Q = \binom{4}{-3}$. Find the slope-intercept form too.
  Solution: $\binom{x}{y} = \binom{1}{1} + s \cdot \binom{3}{-4}$ and $y = -\frac{4}{3} \cdot x + \frac{7}{3}$
- **Example. For the lines**

$$g : \vec{x} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + s \cdot \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} \quad \text{and} \quad h : \vec{x} = \begin{pmatrix} 3 \\ -6 \\ 4 \end{pmatrix} + s \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

  with parameters $s, t \in \mathbb{R}$ find the point of intersection and compute the corresponding angle (which is the angle of the directional vectors)
  Solution: $\begin{pmatrix} 5 \\ -3 \\ 5 \end{pmatrix}$ **and** $\approx 1.90\,\text{rad}$

Planes in Parameter Form

- In $\mathbb{R}^3$ three points $P = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}, Q = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$ and $R = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix}$ determine a **plane**



Planes in Parameter Form

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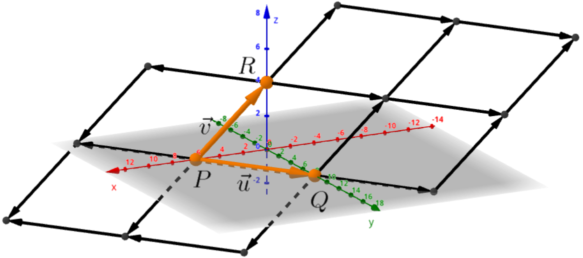
- **The parameter form of the plane is given by**

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = P + s \cdot \vec{u} + t \cdot \vec{v}$$



Planes in Parameter Form

- **How to get the so called plane equation**

$$ax + by + cz = d$$

Planes in Parameter Form

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- **Compute normal vector $\vec{n} = \vec{u} \times \vec{v}$ and**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} + s \cdot \vec{u} + t \cdot \vec{v}$$

Planes in Parameter Form

- How to get the so called plane equation

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- **Compute normal vector $\vec{n} = \vec{u} \times \vec{v}$ and**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = s \cdot \vec{u} + t \cdot \vec{v}$$

Planes in Parameter Form

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$$\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}\right) \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = (s \cdot \vec{u} + t \cdot \vec{v}) \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0$$

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- **Here we see that a plane is also defined by point P and the normal vector of the plane**

Planes in Parameter Form

- How to get the so called plane equation

$$ax + by + cz = d$$

- Compute normal vector $\vec{n} = \vec{u} \times \vec{v}$ and

$$\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} \right) \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0$$

- Here we see that a plane is also defined by point P and the normal vector of the plane
- **But we can also perform the multiplication and get**

$$\underbrace{n_1}_{=a} \cdot x + \underbrace{n_2}_{=b} \cdot y + \underbrace{n_3}_{=c} \cdot z = \underbrace{\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}}_{=d}$$

Planes in Parameter Form

- **Example.** Find the parameter form and the plane equation of the plane with the points

$$P = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad Q = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}, \quad \text{and} \quad R = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

Solution: $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + t \cdot \begin{pmatrix} -2 \\ 0 \\ -6 \end{pmatrix}$, with $s, t \in \mathbb{R}$ and $6x + y + 2z = 2$

Projection

- **Let a point $Q \in \mathbb{R}^n$ and a line $g : \vec{x} = P + s \cdot \vec{v}, s \in \mathbb{R}$ be given. Find the point $\tilde{Q}$ on g such that distance of Q and $\tilde{Q}$ is as small as possible**

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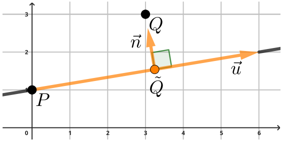
- **We see that we can decompose Q via**

$$Q = \underbrace{P + \lambda \cdot \vec{u}}_{=\tilde{Q}} + \mu \cdot \vec{n}$$



Trick: scalar multiplication by $\vec{u}$:

$$(Q - P) \cdot \vec{u} = (\lambda \cdot \vec{u} + \mu \cdot \vec{n}) \cdot \vec{u} = \lambda \cdot \vec{u} \cdot \vec{u} = \lambda \cdot |\vec{u}|^2$$

Therefore $\lambda = \frac{(Q-P)\cdot\vec{u}}{|\vec{u}|^2}$ **and** $\tilde{Q} = P + \frac{(Q-P)\cdot\vec{u}}{|\vec{u}|^2} \cdot \vec{u}$

$\tilde{Q}$ is called Projection of Q onto g

Projection

- **Attention, this is wrong:**

$$\tilde{Q} = P + \frac{(Q-P)\cdot\vec{u}}{|\vec{u}|^2}\cdot\vec{u} = P + \frac{(Q-P)}{|\vec{u}|^2}\cdot(\vec{u}\cdot\vec{u}) = P + \frac{(Q-P)}{|\vec{u}|^2}\cdot|\vec{u}|^2$$
$$= P + Q - P = Q$$

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- **Example.** **Show**

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \left[\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}\right]\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} \neq \begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\left[\begin{pmatrix}0\\1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right] = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

Which · stand for multiplication scalar times vector? And which · represent the scalar product?

Projection

- **Example.** Find the projection if $Q = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $g : \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Compute the distance $\left| Q - \tilde{Q} \right|$

  Solution: $\tilde{Q} = \begin{pmatrix} 1.5 \\ 0 \\ 0.5 \end{pmatrix}$ and distance ≈ 1.22

Projection

- Same idea but now we want to project onto the subspace

$$U = \{\vec{x} = P + s_1 \cdot \vec{u}_1 + s_2 \cdot \vec{u}_2 + \ldots + s_m \cdot \vec{u}_m\}$$

where P is a point in $\mathbb{R}^n$, parameters $s_i \in \mathbb{R}$ and vectors $\vec{u}_i \in \mathbb{R}^n$ and $m < n$

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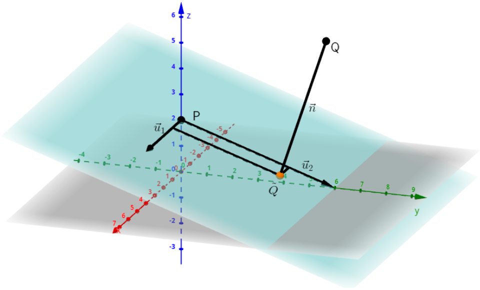

Decomposition: $Q = P + \lambda_1 \cdot \vec{u}_1 + \lambda_2 \cdot \vec{u}_2 + \mu \cdot \vec{n}$

Scalar multiplication with $\vec{u}_1$:

$(Q - P) \cdot \vec{u}_1 = \left(\lambda_1 \cdot \vec{u}_1 + \lambda_2 \cdot \vec{u}_2 + \mu \cdot \vec{n}\right) \cdot \vec{u}_1$

$(Q - P) \cdot \vec{u}_1 = \lambda_1 \cdot \vec{u}_1 \cdot \vec{u}_1 + \lambda_2 \cdot \vec{u}_2 \cdot \vec{u}_1$

Same with $\vec{u}_2$:

$(Q - P) \cdot \vec{u}_2 = \left(\lambda_1 \cdot \vec{u}_1 + \lambda_2 \cdot \vec{u}_2 + \mu \cdot \vec{n}\right) \cdot \vec{u}_2$

$(Q - P) \cdot \vec{u}_2 = \lambda_1 \cdot \vec{u}_1 \cdot \vec{u}_2 + \lambda_2 \cdot \vec{u}_2 \cdot \vec{u}_2$

We get two equations for λ_1 and λ_2 and finally we get projection $\tilde{Q} = P + \lambda_1 \cdot \vec{u}_1 + \lambda_2 \cdot \vec{u}_2$

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  - **Fourier series in signal processing**
  - **Finite element method in numerics**

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- Projections are used very often (without mentioning that it's a projection):
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- **Example. Compute the orthogonal projection for $Q = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ and $U = \left\{ \vec{x} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + s_1 \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + s_2 \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right\}$, with $s_1, s_2 \in \mathbb{R}$.**

  Solution: $\tilde{Q} \approx \begin{pmatrix} 3.33 \\ 2.33 \\ 3.33 \end{pmatrix}$

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- **Example. Use this idea and solve the above example again**

Linear Independence

- **Definition (Linear Independence, Linear Dependence for two Vectors).** Two vectors $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^n$ are linearly independent if the equation

$$\lambda_1 \vec{u}_1 + \lambda_2 \vec{u}_2 = \vec{0}$$

with unknown $\lambda_1, \lambda_2 \in \mathbb{R}$ has only the trivial solution $\lambda_1 = \lambda_2 = 0$. $\vec{0} \in \mathbb{R}^n$ is the "zero vector", i.e. has n zeros. If there exists another solution, then the vectors are called linearly dependent.

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  This is the idea of linear dependence. One vector does not provide a new direction
- **Example. Look at the above example again: Is one vector a multiple of the other vector? What happens if we use these vectors as directional vectors of a plane?**
  Solution: yes, no plane but a line

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- Therefore at least one vector does not provide a new direction and is therefore often unnecessary
- **Very easy and common vectors are the standard unit vectors (here in $\mathbb{R}^4$)**

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

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- **E.g. if $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$ are pairwise orthogonal and $\vec{u}_i \neq \vec{0}$ we get**

$$\lambda_1\vec{u}_1 \cdot \vec{u}_1 + \lambda_2\vec{u}_2 \cdot \vec{u}_1 + \lambda_3\vec{u}_3 \cdot \vec{u}_1 = 0$$

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- E.g. if $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$ are pairwise orthogonal and $\vec{u}_i \neq \vec{0}$ we get

$$\lambda_1 \left|\vec{u}_1\right|^2 = 0 \Rightarrow \lambda_1 = 0$$

  Analogue for $\vec{u}_2$ and $\vec{u}_3 \Rightarrow \lambda_2 = \lambda_3 = 0 \Rightarrow$ linearly independent
- **Standard unit vectors are linearly dependent**

Linear Independence

- **We can decompose each vector $\vec{u} \in \mathbb{R}^n$ by the n standard unit vectors $\vec{e}_i \in \mathbb{R}^n$ via**

$$\vec{u} = s_1 \cdot \vec{e}_1 + s_2 \cdot \vec{e}_2 + \ldots + s_n \cdot \vec{e}_n$$

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- **E.g. for $n = 3$ we have**

$$\begin{pmatrix} 4 \\ 5 \\ 8 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 8 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

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- Standard unit vectors $\vec{e}_i \in \mathbb{R}^n$ span all vectors of $\mathbb{R}^n$
- **Definition (Spanning Property). If a set of vectors $S = \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m\}$ with $\vec{a}_i \in \mathbb{R}^n$ can span all other vectors of $\mathbb{R}^n$, S has the spanning property**

Linear Independence

- **Example.** Does $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ have the spanning property? So can we find λ_1 and λ_2 such that for any vector $\vec{u}$ there holds

$$\vec{u} = \lambda_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Solution: no

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$$\vec{u} = \lambda_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

  Solution: yes

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- **Definition (Basis).** A set of vectors (all in $\mathbb{R}^n$) $S = \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m\}$ a called a basis of $\mathbb{R}^n$ if S has the spanning property and the vectors are linearly independent

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- Standard unit vectors $\vec{e}_i$ are a basis of $\mathbb{R}^n$
- **Example. Which set is a basis of $\mathbb{R}^3$?**

$$S_1 = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}, \qquad S_2 = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\},$$

$$S_3 = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \quad \text{and} \quad S_4 = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}\right\}$$

Solution: S_1 no (no spanning property, too small), S_2 yes (spanning property and independent), S_3 no (not independent and no spanning property), S_4 no (not independent, too large)

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- **Example. Look at the previous example again and use this properties**

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- **If $S = \{\vec{a}_1, \vec{a}_2, \dots, a_n\}$ be a basis of $\mathbb{R}^n$ (we now know that we need n vectors) we can span every vector $\vec{u} \in \mathbb{R}^n$ by**

$$\vec{u} = \lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 + \dots + \lambda_n \vec{a}_n$$

the coefficients $\lambda_i \in \mathbb{R}$ are uniquely determined and are called coordinates of $\vec{u}$ with respect to the basis S and write $\vec{u} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}_S$

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- **If S is the set of the n standard unit vectors $\vec{e}_i$, i.e. $S = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$, we write $\vec{u} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$ and skip S because it is the standard basis**

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- **We know that pairwise orthogonal vectors are linearly independent. Hence, n orthogonal vectors build a basis of $\mathbb{R}^n$ and for any vector $\vec{u} \in \mathbb{R}^n$ we can compute coordinates by**

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- **Example. Compute the coordinates of $\begin{pmatrix} 5 \\ 9 \\ 10 \end{pmatrix}$ with respect to $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$**

  Solution: $\begin{pmatrix} 5 \\ 9 \\ 10 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 10 \end{pmatrix}_S$