Discrete Mathematics and Algebra
CSCI 2025

Session 6

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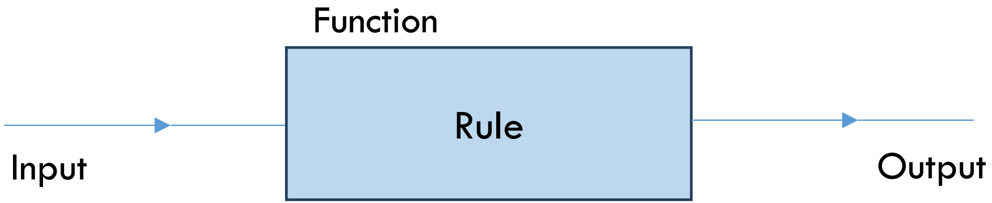
Overview

- Functions
- Linear
- Polynomial
- Exponential

Functions

Introduction

A **function** is a rule that operates on an **input** and produces a **single output** from that input.

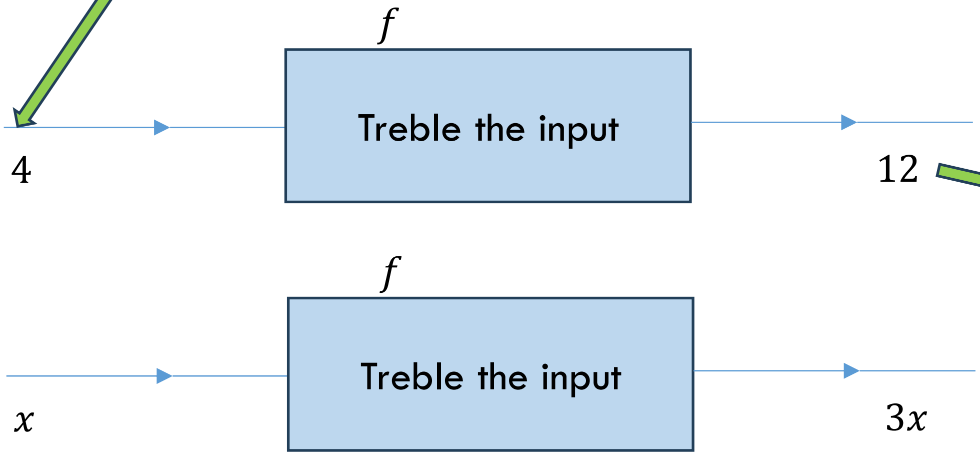


Functions

Introduction

The input to a function is called its **argument.**

For example, the function with rule 'treble the input'.



The value of the output is often called the **value of the function.**

Functions

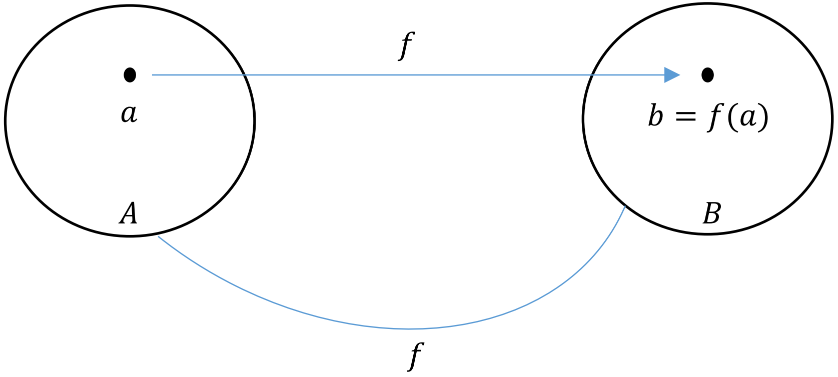
Definition

Let A and B be nonempty sets.

A function f from A to B is an assignment of exactly one element of B to each element of A.

We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

If f is a function from A to B, we write $f: A \to B$.



Functions

Example

Given the function $g(t) = 3t^2 - 7$ find:

- $g(-2)$
- $g(3x)$

Functions

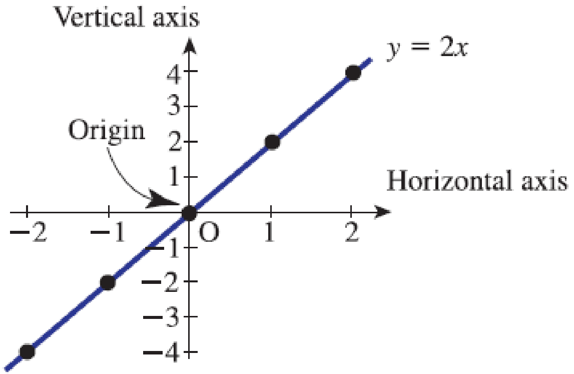
The graph of a function

Consider the function $f(x) = 2x$.

Table of values

Input, x	-2	-1	0	1	2
Output, $f(x)$	-4	-2	0	2	4

Graph



Functions

The domain and range

In the function $y = f(x)$, x is called the **independent variable** and y is called the **dependent variable** because the value of y depends upon the value chosen for x.

- The set of x values used as input to the function is called the **domain** of the function.
- The set of values that y takes as x is varied is called the **range** of the function.

Functions

Example

Consider the function $g(t) = 2t^2 + 1,\ -2 \leq t \leq 2$.

- State the domain of the function.
- Plot a graph of the function.
- Deduce the range of the function from the graph.

Functions

Example

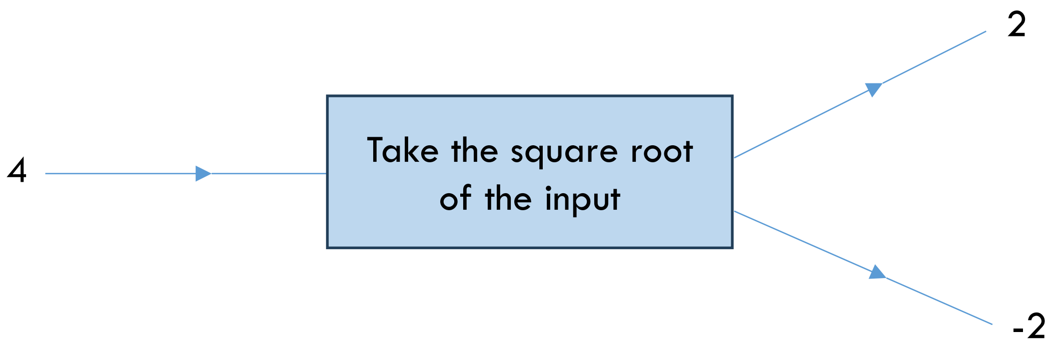
Explain why the values $x = 2$ and $x = -3$ must be excluded from the domain of the function

$f(x) = \frac{5}{(x-2)(x+3)}$.

Functions

One-to-many

One input produces more than one output → This rule **cannot** be a function!



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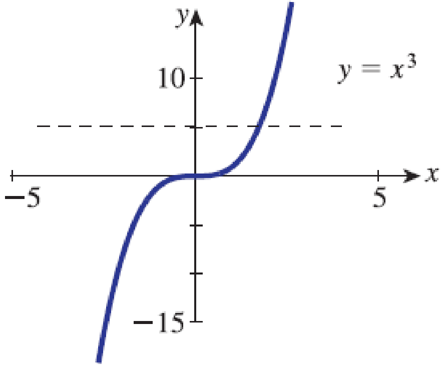
One-to-one

Each different input to the function yields a different output.

The function $y(x) = x^3$ is a one-to-one function.



Functions

Definition

A function f is said to be **one-to-one**, or an **injection**, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be **injective** if it is one-to-one.

Functions

Example

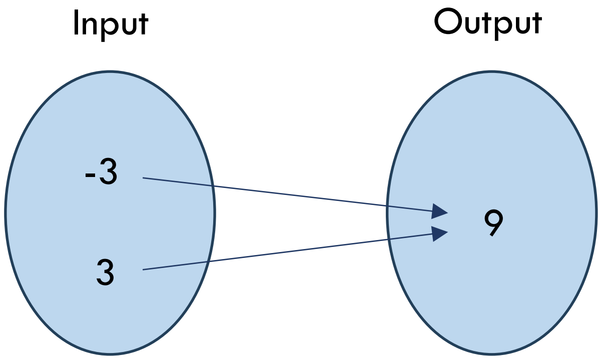
Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f\ (a) = \ 4, f(b) = \ 5,\ \ f\ (c) = 1,$ and $f(d) = \ 3$ is one-to-one.

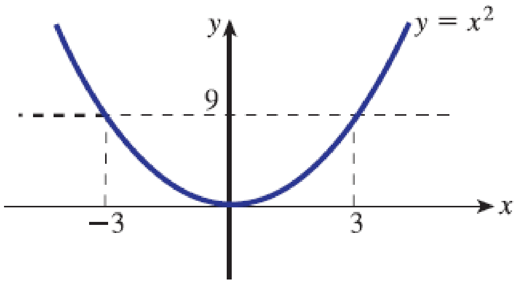
Functions

Many-to-one

Consider the function $y(x) = x^2$.

A function for which different inputs can produce the same output is called a **many-to-one function.**





Functions

Definition

A function f from A to B is called **onto**, or a **surjection**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called **surjective** if it is onto.

Functions

Example

Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3,\ f(b) = 2,\ f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Functions

Definition

The function f is a **one-to-one correspondence**, or a **bijection**, if it is both one-to-one and onto. We also say that such a function is **bijective**.

Functions

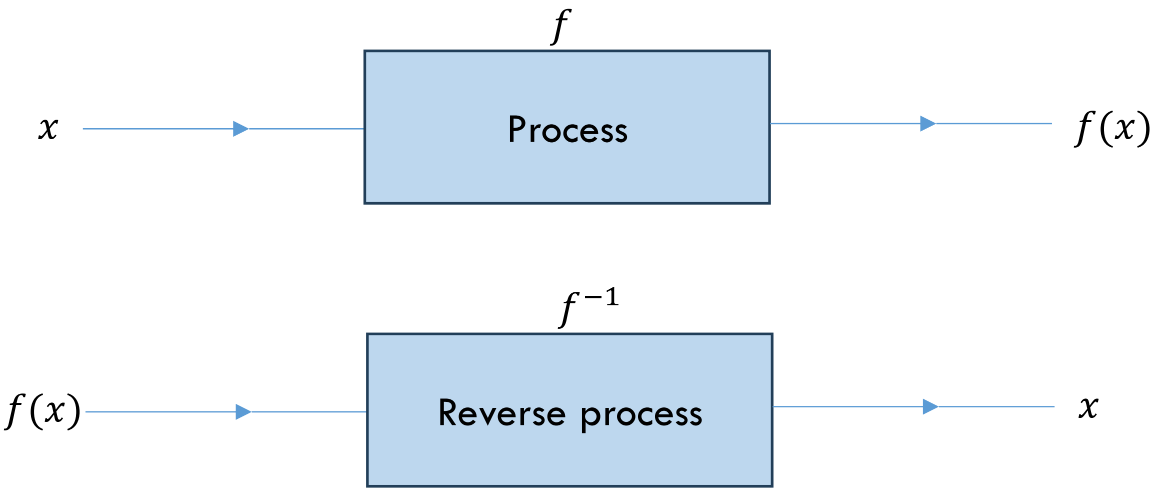
Example

Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ defined by $f(a) = 4,\ f(b) = 2,\ f(c) = 1,$ and $f(d) = 3.$

Is f a bijection?

Functions

Inverse of a function



$f^{-1}(x)$ is the notation used to denote the **inverse function** of $f(x)$.

The inverse function, if it exists, reverses the process in $f(x)$.

Functions

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Definition

Let f be a one-to-one correspondence from the set A to the set B.

The **inverse function** of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$.

The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Functions

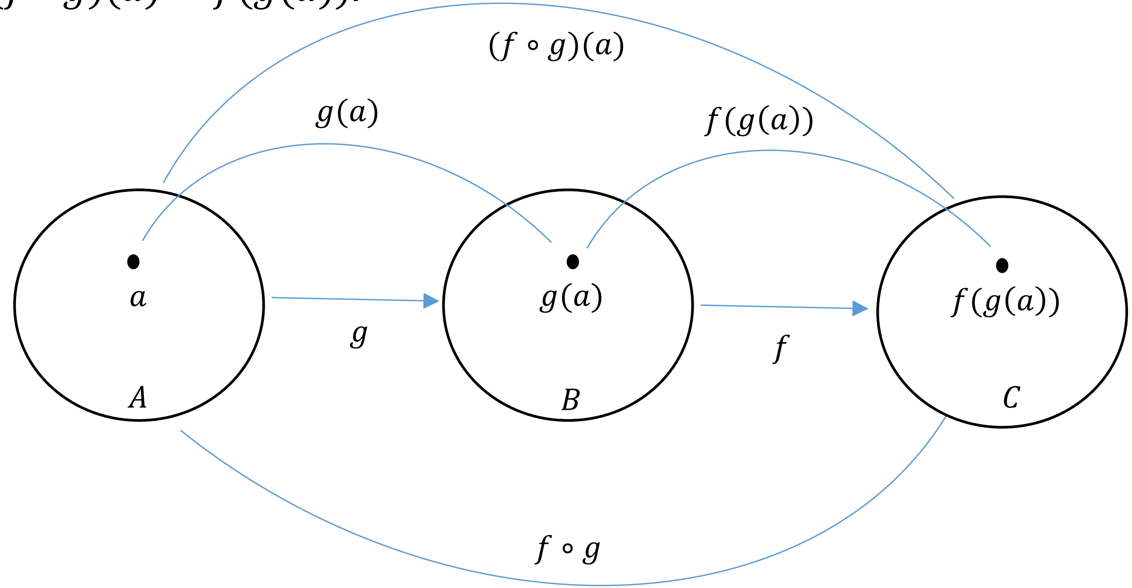
Example

Find the inverse function for $f(t) = 3t - 8$.

Functions

Definition

Let g be a function from the set A to the set B and let f be a function from the set B to the set C.
The **composition** of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is the function from A to C defined by $(f \circ g)(a) = f(g(a))$.



Functions

Example

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.
What is the composition of f and g? What is the composition of g and f?

Functions

Linear functions

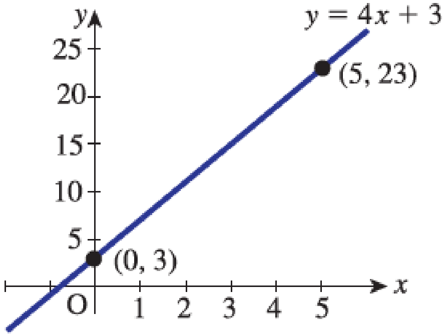
Any function of the form $y = f(x) = ax + b$ is called a linear function.

$\boldsymbol{a}$: coefficient of x

$\boldsymbol{b}$: constant term

Example:

Plot the graph of the linear function $y = f(x) = 4x + 3$.



Functions

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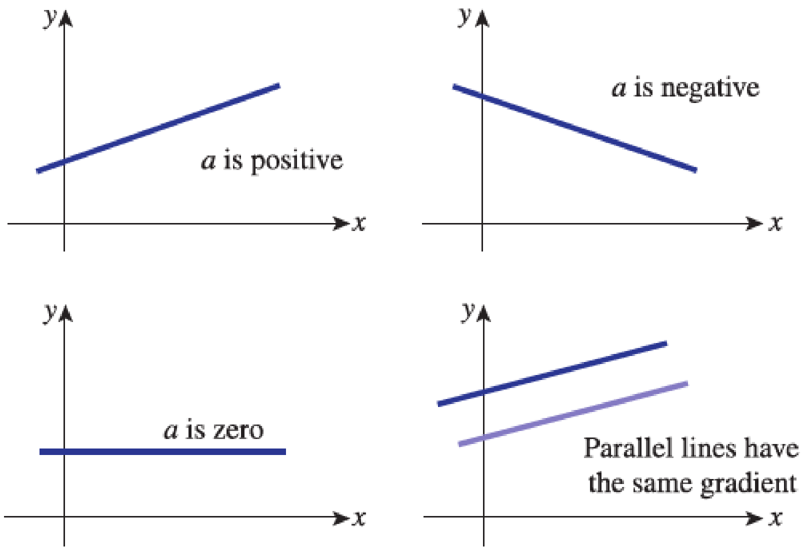
Linear functions

In the linear function $y = ax + b$, $\boldsymbol{a}$ is the **gradient** of the graph and $\boldsymbol{b}$ is its **vertical intercept**.



Functions

Example

Which of the following lines has the steepest gradient?

a) $y = \frac{17x+4}{5}$

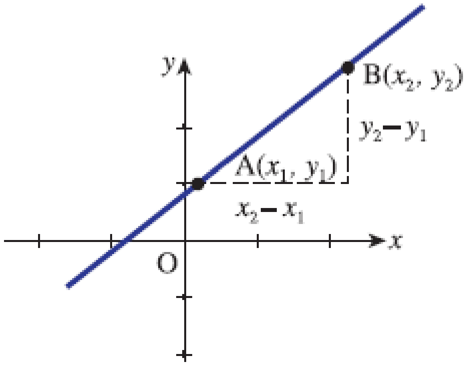
b) $y = 9x - 2$

c) $y = \frac{1}{3}x + 4$

Functions

Gradient

The gradient of the line joining $A(x_1, y_1)$ and $A(x_2, y_2)$ is given by



$$gradient = \frac{vertical\ distance}{horizontal\ diestance}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

Functions

Example

Find the gradient of the line joining each of the following pairs of points:

$a)$ $A(0, 3)$ and $B(4, 5)$

$b)$ $A(-1, 4)$ and $B(2, 1)$

Functions

Example

A straight line has equation $y = 3x + 7$. State, without calculation, the increase in y obtained from a unit increase in x.

Polynomial functions

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A polynomial expression is made up of multiples of non-negative whole number powers of a variable, such as $3x^2$, $-7x^3$, and so on.

Polynomial expressions are used to define polynomial functions. Polynomial functions include:

$$P_1(x) = 3x^2 - x + 2$$
$$P_2(z) = 7z^4 + z^2 - 1$$
$$P_3(t) = 3t + 9$$
$$P_4(x) = 6$$

where x, z and t are independent variables.

Polynomial functions

A **polynomial expression** has the form

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a non-negative integer, $a_n, a_{n-1}, \dots, a_1, a_0$ are constants and x is a variable.

A **polynomial function** $P(x)$ has the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

Polynomial functions

Example

Is $y = 1 + x + x^{1/2}$ a polynomial?

Rational functions

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A rational function is formed by dividing one polynomial by another.

For instance:

$$R_1(x) = \frac{x+2}{x^2+1}$$

$$R_2(t) = \frac{t^3-1}{2t+3}$$

$$R_3(z) = \frac{2z^2+z-1}{z^2+z-2}$$

Rational functions

A rational function has the form

$$R(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomial expressions; P is called the **numerator** and Q is called the **denominator**.

Rational functions

Example

Study the following graph and the algebraic form of the function $R_1(x) = \frac{x+2}{x^2+1}$ carefully and try to answer the following questions.

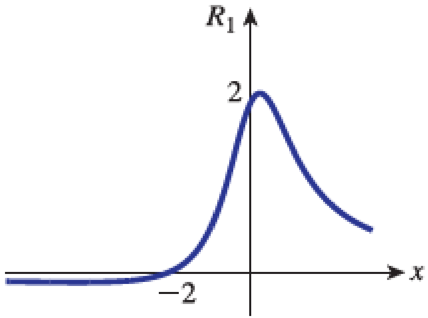
a) For what values of x, if any, is the denominator zero?

b) For what values of x, if any, is the denominator negative?

c) For what values of x is the function negative?

d) What is the value of the function when x is zero?

e) What happens to the function if x gets very large (say 10, 100, …)? Substitute some values to see.



Quadratic equations

A quadratic equation is one that can be written in the form

$$ax^2 + bx + c = 0$$

where a, b and c are numbers and x is the unknown whose value(s) we wish to find.

Quadratic equations

Solution by formula

If

$$ax^2 + bx + c = 0$$

then

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$\Delta = b^2 - 4ac > 0$ two distinct real roots

$\Delta = b^2 - 4ac = 0$ repeated equal root

$\Delta = b^2 - 4ac < 0$ no real roots – there are two distinct complex roots

Example

Use the formula to solve the equations:

- $x^2 + 6x + 5 = 0$
- $4x^2 + 4x + 1 = 0$
- $x^2 - 6x + 10 = 0$

Quadratic equations

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A quadratic equation $ax^2 + bx + c = 0$ can be factorised into an equivalent equation

$$ax^2 + bx + c = a(x - x_1)(x - x_2) = 0$$

where x_1 and x_2 are the solutions for x.

Example:

Factorise the following

- $2x^2 + 3x + 1$
- $5x^2 - 4x - 1$

Exponential functions

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Exponential expression

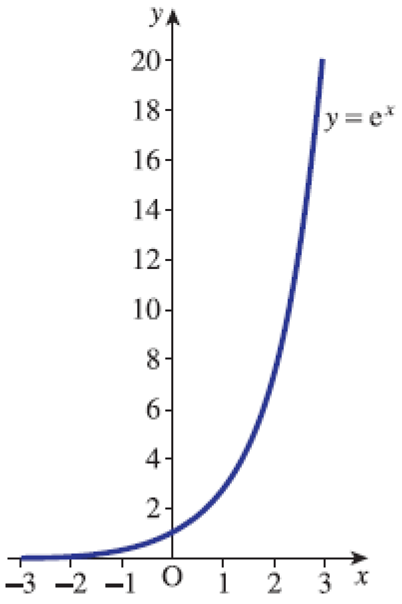
- An exponent is another name for a power or index.
- Expressions involving exponents are called exponential expressions. For instance, 2^3 and a^b.
- In the exponential expression a^x, a is called the base and x is the exponent.
- Laws of indices
  - $a^m \cdot a^n = a^{m+n}$
  - $\frac{a^m}{a^n} = a^{m-n}$
  - $(a^m)^n = a^{mn}$

Exponential function: $y = a^x$

Exponential functions

$y = e^x$

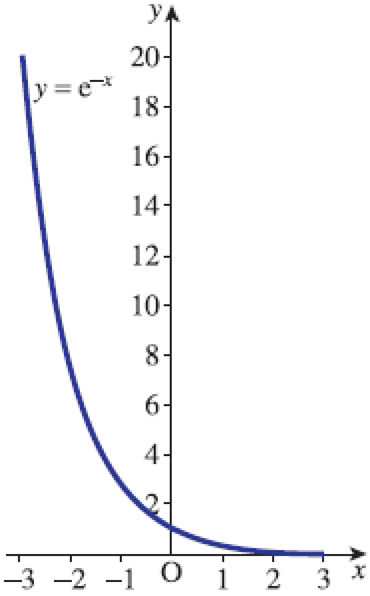
x	e^x	x	e^x
−3.0	0.05	0.5	1.65
−2.5	0.08	1.0	2.72
−2.0	0.14	1.5	4.48
−1.5	0.22	2.0	7.39
−1.0	0.37	2.5	12.18
−0.5	0.61	3.0	20.09
0	1.00		



Exponential functions

$y = e^{-x}$

x	e^{-x}	x	e^{-x}
−3	20.09	0.5	0.61
−2.5	12.18	1.0	0.37
−2.0	7.39	1.5	0.22
−1.5	4.48	2.0	0.14
−1.0	2.72	2.5	0.08
−0.5	1.65	3.0	0.05
0	1.00		



Exponential functions

Example

Plot the graph of the function $y = 3 + 2e^{-x}$.

Which value does y approach as x increases?

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Thank you!