# III. Combinatorics

- Permutations and Combinations



## Permutations and Combinations 1/5

Combinatorics is an area of mathematics primarily concerned with counting.
Understanding the basic ideas is crucial for understanding statistical concepts.

### Permutation without repetition

Factorial of a number

$$n! = n \cdot (n-1) \cdots 2 \cdot 1$$

n! gives the possible arrangements of n distinguishable objects. Note that

$$n! = \prod_{i=1}^{n} i$$

Special cases: $1! = 1$ and $0! = 1$

### Example: Permutation without repetition

**Problem:**
How many different ordered arrangements of the letters a, b, and c are possible?
**Solution:**
There are $3! = 3 \cdot 2 \cdot 1 = 6$ possible arrangements.



## Permutations and Combinations (2/5)

### Permutation with repetition

The number of permutations of n objects with $n_1$ identical objects of type 1, $n_2$ identical objects of type 2, $\cdots$, and $n_k$ identical objects of type k is

$$\frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

### Example: Permutation with repetition

**Problem:**
How many total arrangements of the letters in MISSISSIPPI are there?
**Solution:**

| Letter | Frequency |
|---|---|
| M | 1 |
| I | 4 |
| S | 4 |
| P | 2 |
| total | 11 |

Using the formula above, we get
$\frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = 34650$ possible arrangements.



## Permutations and Combinations (3/5)

### Binomial Coefficient

Binomial Coefficient

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

Note that

The binomial coefficient tells us how many ways there are to choose k things out of larger set consisting of n things.

**Some Theorems:**

Theorem 1

$$\binom{n}{0} = 1$$

Theorem 2

$$\binom{n}{n} = 1$$

Theorem 3

$$\binom{n}{k} = \binom{n}{n-k}$$

Theorem 3 is called the *Symmetry Rule for Binomial Coefficients.*

## Permutations and Combinations (4/5): Example

### Example: Selection problem

**Problem:**
How many teams of 4 people be chosen from a company that employs 20 people?
**Solution:**
There are $\binom{20}{4} = 4845$ ways.

### Note

This is a special case of a permutation with 2 groups (selected vs. non-selected objects).



## Permutations and Combinations (5/5): Examples

### Example 1

**Problem:**
In how many ways can 5 people be seated on a sofa?
**Solution:** 120 ways

### Example 2

**Problem:**
Five red books, two white books and three blue books are arranged in a shelf. If all the books of the same color are not distinguishable from each other, how many different arrangements are possible?
**Solution:** = 2520 arrangements

### Example 3

A teacher selects six exam questions from a pool comprising 15 questions. How many different exams are possible (if the order of the questions is not important)?
**Solution:** = 5005 ways



# IV. Probability (Part I: Basic Concepts)

- Basic Probability with Examples



## Basic Probability with Examples 1/10

**Probability** is a measure of the likelihood of the occurrence of an event. The relative frequency definition of probability states that if an experiment is performed n times, and if event E occurs f times, then the probability of event E is given by the following formula:

Relative frequency definition

$$P(E) = \frac{f}{n}$$

Note that this is only true for **equally likely** outcomes. The **sample space** $\Omega$ of an experiment or random trial is the **set of all possible outcomes** or results of that experiment. An **event** is a subset of the sample space $\Omega$. Note that $\Omega$ itself and the empty set $\{\}$ are also subsets of $\Omega$.

### Example: Relative Frequency Definition

An experiment consists of rolling one unbiased dice. The sample space for this experiment is $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let **E** be the event *an even number occurs*. The event happens, if we get two or four or six $\{2, 4, 6\}$. Since the dice is unbiased (every outcome is equally likely) we get $P(E) = \frac{3}{6} = \frac{1}{2}$ or 50%.

# Basic Probability with Examples 2/10

Range of values for probabilities

$$0 \le P(E) \le 1 \text{ and } P(\Omega) = 1$$

## Example: Range of values

An experiment consists of rolling one unbiased dice. The sample space for this experiment is $\Omega = \{1, 2, 3, 4, 5, 6\}$. Now let us define two events:

- **A** the outcome of the experiment is 7
- **B** the outcome is an integer number

Now we calculate the probabilities: $P(A) = \frac{0}{6} = 0 = 0\%$ and $P(B) = \frac{6}{6} = 1 = 100\%$
Note that **A** is called the **impossible event** and **B** is called the **certain event**.

## Be careful:

Note that an impossible event has zero probability but not all zero-probability events are impossible!



# Basic Probability with Examples 3/10

**Complementary event.** To every event E there is a complementary event $\neg E$.

Rule for the complementary event (very helpful!)

$$P(E) + P(\neg E) = 1$$

## Example: Finding the Probability for a Complementary Event

Approximately 3% of a population is diabetic. The probability that a randomly chosen citizen from that population is not diabetic is 0.97 or 97 %.



# Basic Probability with Examples 4/10

**Multiplication Rules for compound events (A and B occurs): independent events**

Multiplication Rule for independent events

$$P(A \wedge B) = P(A) \cdot P(B)$$

## Example: Finding the probability for A **and** B

Find the probability of rolling a six on a single 6-sided die, **and** then again rolling a six. Since the two experiments are **independent**, we get

$\text{P(six} \wedge \text{six)} = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$



# Basic Probability with Examples 5/10

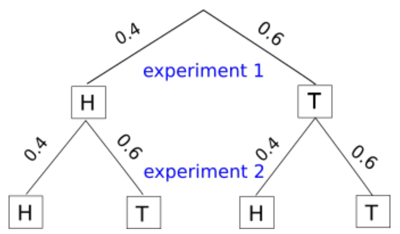
**Drawing a tree diagram.** The best and easiest way to solve problems involving compound events is to use a tree diagram.

- Draw a tree with one branch for each possible outcome in one experiment.
- Write the probability of each branch on the branch and the outcome at the end of the branch.
- Extend the diagram for each rerun of the experiment.
- **Multiply** the probabilities of interest along the branches.
- **Add** the results that lead to the desired outcome.
- If this is done correctly, the final probabilities of **all** branches add up to 1.

## Example:

Experiment: Flip a biased coin twice (probability for head=40%, probability for tail = 60%). Find the following probabilities:

- P(head and head) (Solution: 16%)
- P(at least once head) (Solution: 64%)
- P(two different results) (Solution: 48%)



# Basic Probability with Examples 6/10

**Multiplication Rules for compound events (A and B occurs): dependent events**

**Multiplication Rule for dependent events.** Draw a tree diagram! Note that the probabilities change after each run of the experiment.

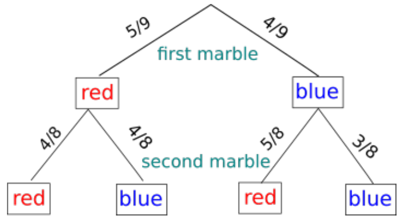
## Example:

A box contains 5 red and 4 blue marbles. Two marbles are withdrawn randomly **without replacement**. Calculate the following probabilities:

- P(both are red) =?

  Solution: $\frac{5}{9} \cdot \frac{4}{8} \approx 27.78\%$
- P(two different colors) =?

  Solution: $\frac{5}{9} \cdot \frac{4}{8} + \frac{4}{9} \cdot \frac{5}{8} \approx 55.56\%$






# Basic Probability with Examples 7/10

**Union of events.** The union of two events A and B consists of all those outcomes that belong to A or B or both A and B.

Addition Rule for the union of events

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$

$\vee$ inclusive OR
$\wedge$ logical AND

If A and B are mutually exclusive, (this means: $P(A \wedge B) = 0$) we get:

$$P(A \vee B) = P(A) + P(B)$$



# Basic Probability with Examples 8/10

## Example

An experiment consists of rolling a die, $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let Z be the outcome. We define two events:

- **A:** Z > 3
- **B:** Z is an even number

Find the probability that A or B (or both) events occur.

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$

$$P(A \vee B) = \frac{3}{6} + \frac{3}{6} - \frac{2}{6} = \frac{2}{3}$$

## Example: mutually exclusive events

**Problem:** A box contains 5 red, 3 green and 4 white marbles. Find the probability that one marble is white or green.
**Solution:** Since the events are mutually exclusive (there are no marbles which are white **and** green). We get: P(white or green) = $\frac{3}{12} + \frac{4}{12} = \frac{7}{12}$.



# Basic Probability with Examples 9/10

## Example 1

**Problem:** One ball is drawn from a bag containing 3 white, 4 red, and 5 black balls. What is the probability that ...

1. it is red?
2. that it is white or red?
3. that it is **not** red?

**Solution: 1.** $\frac{1}{3}$ **2.** $\frac{7}{12}$, **3.** $\frac{2}{3}$

## Example 2

**Problem:** If two dice are tossed, what is the probability of throwing a total of 10 or more? Hint: Tabulate the results!
**Solution:** $\frac{1}{6}$

## Example 3

**Problem:** Alice, Bob, and Christian work independently on a problem. If the respective probabilities that they will solve it are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{2}{5}$, find the probability that the problem will be solved.
**Solution:** 80 %

## Example 4

**Problem** Consider the example of selecting a card at random from an ordindary deck of cards. Find the probability that it is a **King or a Heart**. Hint: There are 52 cards (clubs, diamonds, hearts and spades) and 13 cards each (ace, 2-10, Jack, Queen and King).
**Solution:** $\frac{4}{13}$

## Example 5

**Problem** A coin is tossed and a six-sided die is rolled. Find the probability of getting a head on the coin and a 6 on the die. **Solution:** $\frac{1}{12}$